Representation Theory of Classical Compact Lie Groups

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Dedicated to my family, and to Kim Hyun Hwa

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Contents

Chapter 1

Introduction

In this thesis, we will study the representation theory of compact Lie groups, emphasizing the case of the classical compact groups (namely the groups of orthogonal, unitary and symplectic matrices of a given size). These act (are "represented") as groups of linear symmetries endowing vector spaces with extra structure. Such vector spaces are called representations of the respective group and can be decomposed into subspaces (*irreducible* representations). One of our main goals is to explain the abstract classification of the irreducible representation of classical compact Lie groups.

Although we will not have the opportunity to explore applications of this theory, we must mention that Lie groups and their representation theory can be found in many places in mathematics and physics. For example, the decomposition of the natural representation of the special orthogonal group SO(2) on the space of (say, square-integrable) functions on the circle (or, what amounts to the same, periodic functions on the line), corresponds to the Fourier series decomposition of such functions, while the irreducible representations of SO(3) on the space of functions on $\mathbb{R}^3$ correspond to the "spherical harmonic" functions which are ubiquitous in physics (e. g., in the quantum-mechanical description of hydrogen-like atoms).

This thesis is written with advanced undergraduate mathematics reader in mind. Part of the beauty of Lie groups is that it unites different areas of mathematics together. We will be using several topics from the standard undergraduate mathematics curriculum: linear, abstract algebra, analysis, metric topology, etc. Figuratively speaking, we may think of the study of Lie groups as the center of a wheel and each of the spokes as different branches of mathematics—all meeting together at the center of the wheel.

The classical Lie groups preserve a bilinear form on a real or complex vector space, so we begin by defining bilinear forms and their associated matrix groups in chapter 2. Even if the reader is already familiar with the definitions, we recommend skimming through them to review the notation we will use throughout the rest of the thesis.

In chapter 3, we define the orthogonal, unitary, and symplectic classical groups of matrices, which are perhaps familiar to the reader from linear algebra. For the purposes of this thesis, we call those the classical compact Lie groups. We also define general (abstract) Lie groups as differentiable manifolds with a group operation. Every Lie group has a Lie algebra attached to it, and these algebras will also play an important role in the thesis. It is possible to adopt a Lie algebraic approach to the study of the general aspects of representation theory of Lie groups; however, such approach would hide some (ultimately unavoidable) analytic and topological issues, as well as deny some of the benefits of a more unified approach. For these reasons, we eschew the study of representations of Lie algebras entirely.

In chapter 4 we begin the study of some of the more elementary aspects of the representation theory of compact groups. The normalized Haar integral plays a key role, by allowing to "average" over the group. Our primary interest for this thesis is to decompose representations into irreducible subrepresentations. We next introduce the notion of characters of representations. Characters are very useful tools in understanding representation theory and we will make use of them often.

In chapter 5 we study complex representations of connected abelian Lie groups (tori). Commutativity makes complex irreducible representations one-dimensional. This very important special chapter of the representation theory of compact Lie groups is key to further study of the representations of non-abelian Lie groups.

Chapter 6 revolves about the concept of maximal tori of a Lie group, that is, maximal connected abelian Lie subgroups. In a nutshell, restricting a representation of a compact connected Lie group to a maximal torus thereof does not, in principle, lose any information.

In chapter 7, we study the Lie algebras (and duals thereof) of the maximal tori of classical compact Lie groups. The kernel of the covering map from the Lie algebra of a maximal torus to the torus is called the integer lattice; the latter is intimately related to the overall topology of the Lie group. The dual lattice (in the dual Lie algebra of the maximal torus) to the integer lattice is the *weight lattice* of the group; it plays a crucial role in the classification of irreducible representations.

In the final chapter 8, we learn that weights (elements of the weight lattice) of a connected compact Lie group correspond to irreducible representations. The main goal is to explain how this correspondence is established: the restriction of representations to a maximal torus is key. We also explain a relation between the operation of addition of two weights and the corresponding irreducible representations. We conclude the chapter and thesis with explicit examples that illustrate these correspondences.

Chapter 2

Bilinear Forms

2.1 Bilinear and Sesquilinear Forms

Definitions 2.1. Let V be a vector space over some field $\mathbb{F}$. A *bilinear form* on V is a mapping $B : V \times V \to \mathbb{F}$; it takes two vectors and outputs a real scalar denoted by $B(\cdot\,,\cdot\,)$. Bilinear forms satisfy the property that for all $v, w \in V$, and $\lambda \in \mathbb{F}$,

$$B(\lambda v, w) = \lambda B(v, w) \text{ and } B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w),$$

$$B(v, \lambda w) = \lambda B(v, w) \text{ and } B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2).$$

For all $v \in V$ and $w \in V$, a bilinear form is said to be *symmetric* if $B(v, w) = B(w, v)$, *skew-symmetric* if $B(v, w) = -B(w, v)$, and *nondegenerate* if $B(v, \cdot) = 0$ implies $v = 0$. Nondegeneracy, in other words, means that if we have a nonzero vector v, then there exists some other vector w such that the bilinear form of v and w is nonzero.

If we choose a basis $\mathcal{B} = \{v_1, v_2, ...v_n\}$ of V, we may set up a matrix M that corresponds to the form given by $M = [B]_{\mathcal{B}} = [B(v_i, v_j)]$.

Proposition 2.2. Let B be a bilinear form on a vector space V over $\mathbb{F}$ and $M = [B(v_i, v_j)]$ the matrix of the form with respect to a basis $\mathcal{B}$. Let $[v]_{\mathcal{B}}$ and $[w]_{\mathcal{B}}$ be coordinate vectors of the vectors v and w respectively with basis $\mathcal{B}$. Then

$$B(v, w) = [v]_{\mathcal{B}}^T M[w]_{\mathcal{B}},$$

where the superscript T denotes the matrix transpose.

Definition 2.3. The standard *inner product* $\langle \cdot\,, \cdot \rangle$ of two vectors is an example of a bilinear form where the matrix of the form M with respect to the standard basis is the identity matrix I. Let $v, w \in \mathbb{R}^n$, then $B(v, w) = v^T M w = v^T I w = v^T w = \langle v, w \rangle$.

Proposition 2.4. If B is a bilinear form on V and $M = [B(v_i, v_j)]$ is the matrix of the form, then

i. B symmetric if and only if $M = M^T$,

ii. B skew-symmetric if and only if $M = -M^T$, and

iii. B nondegenerate if and only if M is nonsingular (invertible).

Definitions 2.5. If B is a bilinear form on a vector space V, then B is either

i. positive: $B(v, v) \geqslant 0$, for all $v \in V$,

ii. positive definite: $B(v, v) > 0$ for all $v \in V$ such that $v \neq 0$, or

iii. indefinite: $B(v, v) > 0$ for some $v \in V$, and $B(w, w) < 0$ for some $w \in V$.

Definition 2.6. Let $\mathbb{F}$ be a field. The standard nondegenerate alternating bilinear form on $\mathbb{F}^{2n}$ is a bilinear form whose matrix (with respect to the standard basis) is $J_{2n} = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$, where I_n is the $n \times n$ identity matrix. The inner product between two vectors v and w for the symplectic form is $\langle v, w \rangle = v^T J w$.

J_{2n} is an example of the simplest skew-symmetric matrix.

Definitions 2.7. If V is a vector space over $\mathbb{C}$, the set of complex numbers, there is a mapping analogous to the bilinear form, $H : V \times V \to \mathbb{C}$ such that for all $v \in V$, $w \in V$, and $\mu \in \mathbb{C}$,

$$H(\mu v, w) = \bar{\mu} H(v, w) \text{ and } H(v_1 + v_2, w) = H(v_1, w) + (v_2, w),$$

$$H(v, \mu w) = \mu H(v, w) \text{ and } H(v, w_1 + w_2) = H(v, w_1) + (v, w_2).$$

In this case we say that H is a *sesquilinear form* on V. Sesquilinear forms are *skew-linear* or *conjugate-linear* on the first variable, and linear on the second variable. As with the bilinear form, a Sesquilinear form can be symmetric, skew-symmetric, and/or nondegenerate. If H has the property $H(v, w) = \overline{H(w, v)}$, then H is called *Hermitian.*

Remark 2.8. If H is Hermitian, then $H(v, v)$ is a real number for all $v \in V$.

Proposition 2.9. Let H be a sesquilinear form on a vector space V over $\mathbb{C}$ and $M = [B(v_i, v_j)]$ the matrix of the form with respect to a basis $\mathcal{B} = \{v_1, v_2, ..., v_n\}$. Let $[v]_{\mathcal{B}}$ and $[w]_{\mathcal{B}}$ be coordinate vectors of the vectors v and w respectively with basis $\mathcal{B}$. Then

$$H(v, w) = [v]_{\mathcal{B}}^* M [w]_{\mathcal{B}},$$

where the superscript $*$ denotes the conjugate transpose, i.e., $[v]_{\mathcal{B}}^* = \overline{[v]_{\mathcal{B}}}^T$.

Definition 2.10. We define the (standard) *Hermitian inner product* of two vectors $v, w \in \mathbb{C}^n$ to be $\langle v, w \rangle = v^* w$.

We have discussed three types of inner products so far. Henceforth we will refer to each of these (and others) simply as *inner products.* It should be clear from context about the type of inner product that is being used.

Proposition 2.11. If H is a sesquilinear form on V and $M = [H(v_i, v_j)]$ is the matrix of the form, then

i. H symmetric if and only if $M = M^T$,

ii. H skew-symmetric if and only if $M = -M^T$,

ii. H Hermitian if and only if $M = \overline{M}^T = M^*$, and

iii. H nondegenerate if and only if M is nonsingular (invertible).

Definitions 2.12. If H is a Hermitian form on a vector space V, then H is either

i. positive: $H(v, v) \geqslant 0$, for all $v \in V$,

ii. positive definite: $H(v, v) > 0$ for all $v \in V$ such that $v \neq 0$, or

iii. indefinite: $H(v, v) > 0$ for some $v \in V$, and $H(w, w) < 0$ for some $w \in V$.

Remark 2.13. If the Hermitian form H is positive definite then it is also nondegenerate.

Chapter 3

Classical Lie Groups

3.1 The Orthogonal Group

Definition 3.1. The general linear group, $\mathrm{GL}(n, \mathbb{R})$, is the set of all $n \times n$ invertible matrices with real entries. The *orthogonal group* of degree n, $\mathrm{O}(n)$, is defined as the set of all real invertible $n \times n$ matrices preserving the standard inner product, i.e.,

$$\mathrm{O}(n) = \{g \in GL(n, \mathbb{R}) : \langle gv, gw \rangle = \langle v, w \rangle\}.$$

Since $\langle gv, gw \rangle = (gv)^T I(gw) = v^T(g^T g)w$ and $\langle v, w \rangle = v^T I w$, we may also define $O(n)$ as

$$\mathrm{O}(n) = \{\ g \in GL(n, \mathbb{R}) : g^T g = I\ \}.$$

The orthogonal group has several interesting properties. For instance,

Proposition 3.2. A matrix g is orthogonal if and only if $||gv|| = ||v||$, for all $v \in \mathbb{R}^n$

(Note: $||\cdot||$ above stands for the length of a vector, namely $||\cdot|| = \sqrt{\langle\cdot,\cdot\rangle}$).

Proof. Suppose g is orthogonal. Then by definition $||gv|| = \langle gv, gv\rangle = \langle v, v\rangle = ||v||$ for all $v \in \mathbb{R}^n$. Now suppose $\langle gv, gv\rangle = \langle v, v\rangle$. Then by the Polarization Identity,

$$\langle gv, gw\rangle = \frac{||g(v+w)||^2 - ||gv||^2 - ||gw||^2}{2} = \frac{||v+w||^2 - ||v||^2 - ||w||^2}{2} = \langle v, w\rangle$$

hence g is orthogonal. □

Proposition 3.3. Every (complex) eigenvalue of $g \in \mathrm{O}(n)$ has length 1.

Proof. Suppose λ is an eigenvalue of $g \in \mathrm{O}(n)$. Then for some eigenvector v,

$$gv = \lambda v. \tag{3.1}$$

Taking the conjugate transpose of (3.1) gives

$$(gv)^* = (\lambda v)^*,$$

or

$$v^* g^* = v^* \lambda^* = v^* \bar{\lambda} = \bar{\lambda} v^*. \tag{3.2}$$

Now, left multiplying (3.1) with (3.2) gives us

$$v^* g^* g v = \bar{\lambda} v^* \lambda v.$$

Simplifying with $g^* g = I$ gives

$$v^* v = \bar{\lambda}\lambda v^* v,$$

or equivalently,

$$|v|^2 = |\lambda|^2 |v|^2.$$

Thus $|\lambda|^2 = 1$ and $|\lambda| = 1$. □

Definition 3.4. The *special orthogonal group* $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ of matrices with determinant 1 and $\mathrm{O}_{-}(n) \subset \mathrm{O}(n)$ is the set of matrices with determinant -1.

Proposition 3.5. $\mathrm{O}(n)$ is the union of $\mathrm{SO}(n)$ and $\mathrm{O}_{-}(n)$.

Proof. If $g \in \mathrm{O}(n)$, then $g^T g = I$. Taking determinants gives $\det(g^T g) = \det I$, i.e., $\det g^T \det g = (\det g)^2 = 1$. Thus the matrices of $\mathrm{O}(n)$ has determinant either 1 or -1, and so it is the union of $\mathrm{SO}(n)$ and $\mathrm{O}_{-}(n)$. □

Now we will parametrize the group $\mathrm{O}(2)$ to find a general form of a 2×2 orthogonal matrix. Let $g \in \mathrm{O}(2)$. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{R}.$$

Then, since g is orthogonal,

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By matrix multiplication, we have that $a^2 + c^2 = 1 = b^2 + d^2$ and $ab + cd = 0$. (a, c) and (b, d) are points on the unit circle, hence of the form $(\cos\alpha, \sin\alpha)$ and $(\cos\beta, \sin\beta)$ for $\alpha, \beta \in [0, 2\pi)$. Then $0 = ab + cd = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$. Now $\alpha - \beta = \pm\pi/2$ hence $\beta = \alpha \pm \pi/2$.

If $\beta = \alpha + \pi/2$, then $b = \cos\beta = -\sin\alpha$ and d $= \sin\beta = \cos\alpha$, so g will be of the form

$$g = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \alpha \in \mathbb{R},$$

which happens to be the matrix of a rotation! This is the group SO(2).

Now, if we had chosen $\beta = \alpha - \pi/2$ instead, then $b = \cos\beta = \sin\alpha$ and d $= \sin\beta = -\cos\alpha$, so g will be of the form

$$g = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \qquad \alpha \in \mathbb{R},$$

which is the matrix of a reflection over the slope $\tan\theta/2$. The set of matrices of this form is O₋(2).

Proposition 3.6. SO(2) is abelian. However, SO(n) for $n > 2$ is nonabelian.

Proof. The elements of SO(n) for $n > 2$ act by matrix multiplication, which is in general non commutative. However, SO(2) is just the set of one-dimensional rotations on the plane, and the order in which the plane is rotated is commutative. □

Proposition 3.7. *$O(n)$ is a compact topological subspace of the space $\mathfrak{gl}(n, \mathbb{R})$ of all $n \times n$ real matrices. (Note that $\mathfrak{gl}(n, \mathbb{R})$ is a euclidean topological space isomorphic to $\mathbb{R}^{n^2}$, so it has a natural topology.)*

Proof. By the Bolzano-Weierstrass Theorem, we need only show that $O(n)$ is closed and bounded. Let g_{jk} be the entries of an $n \times n$ orthogonal matrix g for $1 \leq j, k \leq n$. Each column of g is orthonormal, i.e., $\sum_{j=1}^{n} g_{jk}^2 = 1$, so the norm $||g|| = \sqrt{\sum_{k=1}^{n}\sum_{j=1}^{n} g_{jk}} = \sqrt{n}$, thus $O(n)$ is bounded. To show closure, it suffices to take a convergent sequence $\{g_m\} \subset \mathrm{O}(n)$ and show that $g = \lim_{n\to\infty} g_m \in \mathrm{O}(n)$. From $g_m^T g_m = I$, we wish to show $g^T g = I$. Let $(g_m)_{ij}$ be the entries of g_m. Then $(g_m^T)_{ij}(g_m)_{ij} = (g_m)_{ji}(g_m)_{ij} = I$ implies $\sum_k^n (g_m)_{ki}(g_m)_{kj} = \delta_{ij}$, where δ_{ij} is 1 for $i = j$ and 0 for $i \neq j$. Taking products or additions of sequences are continuous functions, so the limit as $n \to \infty$ for each of the equations (where i, j varies) is $\sum_k^n g_{ki} g_{kj} = \delta_{ij}$, i.e., $(g^T)_{ij}(g)_{ij} = \delta_{ij}$, or equivalently, $g^T g = I$.

□

3.2 The Unitary Group

Definitions 3.8. The *unitary group* of degree n is the set of all complex invertible $n \times n$ matrices preserving the standard positive definite Hermitian form H of $\mathbb{C}^n$, i.e.,

$$U(n) = \{g \in GL(n, \mathbb{C}) : H(gv, gw) = H(v, w)\},$$

or equivalently,

$$U(n) = \{\ g \in GL(n, \mathbb{C})\ :\ g^* g = I\ \}.$$

U(n), is the Hermitian counterpart to O(n).

Definition 3.9. The *special unitary group* SU(n) is the subgroup of U(n) consisting of matrices of determinant 1.

Example 3.10. Consider the unitary group of 1×1 matrices, U(1). Each $g \in$ U(1) has the property that $g^* g = 1$, hence U(1) is the set of all complex numbers with length 1, so each $g \in U(1)$ is of the form $e^{i\alpha}$, for $\alpha \in \mathbb{R}$. The set of the elements of U(1) from the unit circle on the complex plane $\mathbb{C}$. Left multiplying some $h \in U(1)$ with g simply rotates g on the unit circle on the complex plane.

Remark 3.11. U(1) is isomorphic to SO(2) by the mapping

$$e^{i\alpha} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Proposition 3.12. U(n) is compact.

Proof. The proof is essentially identical to that of O(n). □

3.3 The Symplectic Group

Definitions 3.13. The *standard symplectic (sesquilinear) form* on a vector space $\mathbb{C}^{2n}$, $A(v, w)$, is given by

$$A(v, w) = v^T J w, \quad \text{where } J = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$$

and I_n denotes the identity matrix.

We define the *(complex) symplectic group of degree* $2n$ over $\mathbb{C}$, $Sp(2n, \mathbb{C})$ as the set of elements of $GL(2n, \mathbb{C})$ that leave the standard symplectic form invariant, i.e.,

$$Sp(2n, \mathbb{C}) = \{g \in GL(2n, \mathbb{C}) : A(gv, gw) = A(v, w)\}$$

Or alternatively, since $A(gv, gw) = (gv)^T J(gw) = v^T(g^T Jg)w$ and $A(v, w) = v^T Jw$, we may define $Sp(2n, \mathbb{C})$ as the set

$$Sp(2n, \mathbb{C}) = \{g \in GL(2n, \mathbb{C}) : g^T Jg = J\}.$$

Proposition 3.14. The complex symplectic group is noncompact.

Proof. It suffices to show $\mathrm{Sp}(2,\mathbb{C})$ is unbounded. The matrix

$$s = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$$

is an element of $\mathrm{Sp}(2,\mathbb{C})$, since

$$s^T J s = sJs$$

$$= \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = J$$

Since the norm of s is $||s|| = \sqrt{a^2 + a^{-1}} > |a|$, $\mathrm{Sp}(2,\mathbb{C})$ is unbounded.

□

Restricting $Sp(2n,\mathbb{C})$ to consist of only unitary matrices, we get the *unitary symplectic group*, which we will denote as $USp(2n)$,

$$USp(2n) = \{g \in U(2n) : g^T J g = J\}.$$

Proposition 3.15. $USp(2n)$ is compact.

Proof. $USp(2n) = U(2n) \cap Sp(2n,\mathbb{C})$. Hence $USp(2n) \subset U(2n)$, which is compact. □

For the remainder of this thesis we will be working exclusively with the unitary symplectic group (for its compactness) and will henceforth refer to the unitary symplectic group as *the* symplectic group.

More on the unitary and symplectic group can be found in the first chapter of [Ch].

3.4 Lie Groups

Definition 3.16. A *(topological) manifold* M of dimension m is a topological space which is locally euclidean of dimension m: M is covered by a family F of (not necessarily disjoint) open sets, each homeomorphic to an open set in $\mathbb{R}^m$. Such local homeomorphisms are called *(local) parametrizations*, since they attach coordinate m-tuples to points in each of the open sets in F.

The manifold M is called *differentiable* (or *smooth*) if, whenever two open sets in F intersect, the corresponding coordinates (on the intersection) are related by an (infinitely) differentiable invertible mapping between the corresponding neighborhoods in $\mathbb{R}^m$.

An important result known as the Whitney Embedding Theorem shows that any differentiable manifold M is"diffeomorphic" to a differential submanifold $\tilde{M}$ embedded in euclidean space $\mathbb{R}^k$. In practice, this means that a differentiable manifold can be assumed to be a locally euclidean smooth subset of $\mathbb{R}^k$ such that the local dimension is always the same number m (the dimension of the manifold), and moreover that coordinate systems on M can be obtained by taking (some of the) coordinate functions of the ambient space $\mathbb{R}^k$.

Henceforth we will often omit the adjective "differentiable" in favor of just "manifold", but the smoothness will always be assumed.

For a more precise definition of a manifold, see the first chapter of [dC].

Definition 3.17. A *Lie group* G is a differentiable manifold endowed with

1. a binary operation (group multiplication) $G \times G \to G$ with respect to which G is a group, and

2. the corresponding inversion mapping $inv : G \to G$, $g \mapsto g^{-1}$

both of which are (infinitely) differentiable maps.

Examples 3.18. $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{USp}(2n)$ are Lie groups.

Proof. We will only prove that $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{USp}(2n)$ are subgroups of the Lie group $\mathrm{GL}(n, \mathbb{C})$. Consider first $\mathrm{U}(n)$. Clearly $\mathrm{U}(n)$ has the identity element I, since $I^*I = I$. Suppose $g, h \in \mathrm{U}(n)$. Now we show that h has an inverse h^{-1} in $\mathrm{U}(n)$: Since $h^{-1} = h^*$, $(h^{-1})^*(h^{-1}) = h^{**}h^* = hh^* = h^*h = I$. Now it suffices to show that $\mathrm{U}(n)$ is closed under multiplication: $(gh)^*(gh) = h^*g^*gh = h^*Ih = h^*h = I$. So by the Two-Step Subgroup Test, $\mathrm{U}(n)$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$. It follows that $\mathrm{O}(n)$ and $\mathrm{USp}(2n)$ are also subgroups of $\mathrm{GL}(n, \mathbb{C})$, since they are subgroups of $\mathrm{U}(n)$. $\square$

$\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{USp}(2n)$ are a few of the Lie groups that make up what is known as the "classical groups." They are the set of transformations in $\mathrm{Aut}(V)$ (the set of automorphisms on V) that preserve a bilinear form. These groups are compact, connected, and they are generally nonabelian.

3.5 Lie Algebras

Definition 3.19. A *Lie algebra* $\mathfrak{g}$ is a vector space over $\mathbb{R}$ with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*. In addition to bilinearity, the Lie bracket is skew-symmetric (i.e., $[X,Y] = -[Y,X]$), and also satisfies the *Jacobi identity*:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

For $\mathrm{GL}(n,\mathbb{C})$ and its subgroups (matrix groups), the Lie bracket of their Lie algebra is given by $[X,Y] = XY - YX$. Since all of the groups that we will be studying are subgroups of $\mathrm{GL}(n,\mathbb{C})$, this is the only bracket we will be using.

We will denote Lie algebras of G by $\mathrm{Lie}(G)$, $\mathrm{L}(G)$, or by the corresponding lowercase German Fraktur letter, e.g., $\mathrm{Lie}(G) = \mathfrak{g}$.

Lie algebras are vector spaces! This reduces difficult group theory problems into linear algebra problems. However, we will not focus on Lie algebras in length, aside from a few examples and some essential properties that will be used in later chapters.

Example 3.20. Consider the group $\mathrm{U}(n)$. Let $g : (a,b) \to \mathrm{U}(n)$ be a differentiable curve going through the origin of $\mathrm{U}(n)$ such that $g(0) = I$ where I is the identity matrix. Since for $a < t < b$, $g(t)$ is in $\mathrm{U}(n)$,

$$g(t)^* g(t) = I.$$

Taking derivatives with respect to t gives

$$\frac{d}{dt}[g(t)^* g(t)] = \frac{d}{dt}(I),$$

$$\frac{d}{dt}(g(t)^*)g(t) + g(t)^* \frac{d}{dt} g(t) = O,$$

(here we are using O to denote the 0-matrix)

$$\frac{d}{dt}(g(t))^* g(t) + g(t)^* \frac{d}{dt} g(t) = O,$$

Now we take $t = 0$ and define $X = \frac{d}{dt}(g(0)) \in \mathfrak{u}(n)$ so that

$$X^* I + I^* X = O,$$

$$X^* + X = O,$$

$$X^* = -X.$$

Thus we may define $\mathfrak{u}(n)$ as the set of all $n \times n$ skew-Hermitian matrices, i.e.,

$$\mathfrak{u}(n) = \{X_{n\times n} : X^* = -X\}.$$

To show closure in $\mathfrak{u}(n)$ with respect to the Lie bracket, take X and Y in $\mathfrak{u}(n)$ and show that $[X, Y]$ is in $\mathfrak{u}(n)$, i.e., $[X, Y]$ is skew-Hermitian. Now $[X, Y]^* = (XY - YX)^* = Y^*X^* - X^*Y^* = (-Y)(-X) - (-X)(-Y) = YX - XY = -(XY - YX) = -[X, Y]$.

Similarly as in the previous example, we may define the Lie algebra of $O(n)$, as the set of all skew-symmetric matrices:

$$\mathfrak{o}(n) = \{X_{n\times n} : X^T = -X\},$$

and the Lie algebra of USp$(2n)$ as

$$\mathfrak{usp}(2n) = \{X_{2n\times 2n} : X^* = -X \text{ and } X^T J = -JX\},$$

Proposition 3.21. The dimension of $\mathfrak{u}(n)$ is n^2.

Proof. Since $\mathfrak{u}(n)$ is the space of matrices X such that $X^* = -X$, then X must be of the form

$$\begin{pmatrix} ia_1 & & Z \\ & \ddots & \\ -Z^* & & ia_n \end{pmatrix}$$

for all $a_p \in \mathbb{R}, 1 \leq p \leq n$ and where Z is the upper triangular consisting of complex numbers (note: $-Z^*$ is just the negative complex conjugate transpose of Z, so the entries there do not supply extra dimensions).

Now, Let D be a diagonal matrix with real entries and let T be an upper triangular complex matrix with 0's in the diagonal. Then $X = iD + T - T^*$. There are $1 + \cdots (n-1) = \frac{n(n-1)}{2}$ complex entries in T, which provide $n(n-1)$ real dimensions, and adding the n dimensions from the diagonal matrix D gives a total of $n(n-1) + n = n^2$ dimensions. $\square$

Proposition 3.22. The dimension of $\mathfrak{su}(n)$ is $n^2 - 1$.

Proof. Since $\mathfrak{su}(n)$ has one extra restriction $\mathrm{Tr}(X) = 0$ from $\mathfrak{u}(n)$, the dimension of $\mathfrak{su}(n)$ is simply $\dim(\mathfrak{u}(n)) - 1 = n^2 - 1$. $\square$

Proposition 3.23. The dimension of $\mathfrak{so}(n)$ is $\frac{n(n-1)}{2}$.

Proof. $\mathfrak{so}(n)$ is the space of matrices X such that $X^T = -X$, so X must be of the form

$$\begin{pmatrix} 0 & & A \\ & \ddots & \\ -A^T & & 0 \end{pmatrix}$$

The diagonal consists of 0's and there are $1 + \cdots + (n-1) = \frac{n(n-1)}{2}$ real entries in the upper triangular A (the lower triangular $-A^T$ provides no extra information). Hence the dimension of $\mathfrak{so}(n)$ is $\frac{n(n-1)}{2}$. □

Proposition 3.24. The dimension of $\mathfrak{usp}(2n)$ is $2n^2 + n$.

Proof. $\mathfrak{usp}(2n)$ is the space of matrices X such that

(i.) $X^* = -X$ and

(ii.) $X^T J = -JX$.

Let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are complex block matrices.

First, to satisfy (i.),

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} -A & -B \\ -C & -D \end{pmatrix}$$

which shows that A and D are anti-Hermitian and $C = -B^*$.

Lastly, to satisfy (ii.), we have

$$\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} & -I \\ I & \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

which shows that C and B are symmetric and $D = -A^T$.

Hence X is of the form

$$X = \begin{pmatrix} A & B \\ -B^* & -A^T \end{pmatrix}$$

where $A \in \mathrm{U}(n)$ and B is in the space of complex symmetric matrices. Hence A has real dimension n^2 and B has real dimension $n(n+1)$; giving a total of $n^2+n(n+1) = 2n^2+n$. □

Proposition 3.25. The dimension of a Lie group is the same as the dimension of its Lie algebra.

Proof. This follows from the fact that the tangent space at any point of the manifold has the same dimension as the manifold. See [GiPo], section 1.2. □

Definition 3.26. A *one-parameter subgroup* of a Lie group G is a differentiable homomorphism $\phi : \mathbb{R} \to G$ from the (additive) Lie group $\mathbb{R}$ into G.

Proposition 3.27. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there exists a unique map $\mathrm{Exp} : \mathfrak{g} \to G$ with the property that for any $X \in \mathfrak{g}$, $\mathrm{Exp}(tX)$ is the unique one-parameter subgroup of G with tangent vector X at the identity.

Proof. See [Ar], theorem 9.5.2. □

Definition 3.28. $\mathrm{Exp} : \mathfrak{g} \to G$ is called the *exponential map*.

Proposition 3.29. The exponential map has the following properties:

(i.) $\mathrm{D}\,\mathrm{Exp}_0$ is the identity map from $\mathfrak{g} \to \mathfrak{g}$.

(ii.) For any Lie group homomorphism $\phi : G \to H$, the following diagram commutes:

$$\begin{array}{rccc} \phi: & G & \longrightarrow & H \\ & \operatorname{Exp}\uparrow & & \uparrow \operatorname{Exp} \\ \mathrm{D}\,\phi_0: & \mathfrak{g} & \longrightarrow & \mathfrak{h} \end{array}$$

Proof. $\mathrm{D}\operatorname{Exp}_0 : \mathrm{D}\,\mathfrak{g}_0 \to \mathrm{D}\,G_e = \mathfrak{g}$. Since $\mathfrak{g}$ is the tangent space at the identity of G (3.27), the differential of the tangent space at 0 is the tangent space itself. Hence $\mathrm{D}\operatorname{Exp}_0 : \mathfrak{g} \to \mathfrak{g}$. This proves (i.).

For (ii.), see [Ad], proposition 2.11. □

Proposition 3.30. If G is a matrix group, then its Lie algebra $\mathfrak{g}$ is a matrix Lie algebra, and

$$\operatorname{Exp}(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \text{ for all } X \in \mathfrak{g}\,.$$

Proof. We wish to show that for all $t \in \mathbb{R}$, $\phi(t) = \operatorname{Exp}(tX)$ is a one-parameter subgroup of G with $\phi'(t) = X$ when $t = 0$. Since tX and sX commute for all $t, s \in \mathbb{R}$, we have $\phi(t+s) = \operatorname{Exp}(tX + sX) = \operatorname{Exp}(tX)\operatorname{Exp}(sX) = \phi(t)\phi(s)$, which shows $\operatorname{Exp}(tX)$ is a homomorphism. Furthermore, $\phi'(0) = \mathrm{D}\operatorname{Exp}(tX)|_{t=0} = X\operatorname{Exp}(0) = X$. □

Example 3.31. We will show that the exponential mapping carries elements of $\mathfrak{so}(2)$ to $\mathrm{SO}(2)$. Take $X \in \mathfrak{so}(2)$. Then X is of the form $\begin{pmatrix} & -a \\ a & \end{pmatrix}$ for $a \in \mathbb{R}$. Now the exponential map of X gives:

$$e^X = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} & -a \\ a & \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -a^2 & \\ & -a^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} & a^3 \\ -a^3 & \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} a^4 & \\ & a^4 \end{pmatrix} + \frac{1}{5!}\begin{pmatrix} & -a^5 \\ a^5 & \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \frac{a^2}{2!}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \frac{a^4}{4!}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \mp \cdots + a\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} - \frac{a^3}{3!}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} + \frac{a^5}{5!}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \mp \cdots$$

$$= \cos a \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \sin a \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$$= \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}, \qquad \text{which is an element of } SO(2).$$

Chapter 4

Elementary Representation Theory

In this chapter we will explore a few basic properties of representations of compact groups on a finite vector space (nothing here depends on the connectedness). Some of the properties below do not hold for noncompact groups and so the representation theory for noncompact groups can be more complicated. For this thesis, we will only be working with finite dimensional representations. In addition to a purely algebraic homomorphism property, representations of Lie groups must also satisfy a differentiability property, as explained in definition 4.6.

4.1 The Normalized Haar Integral on a Compact Group

Definition 4.1. A group is *compact* if its group operation $G \times G \to G$, $(g, h) \mapsto gh$ and inversion map $G \to G$, $g \mapsto g^{-1}$ are continuous.

Theorem 4.2. Let G be a compact group and let $C(G, \mathbb{R})$ be the set of continuous functions $f : G \to \mathbb{R}$. Then there exists a unique bi-invariant normalized "integral", that is, a linear functional $I : C(G, \mathbb{R}) \to \mathbb{R}$, where we denote $I(f)$ by $\int_G f(g) d_H g$ (Note: the H stands for "Haar"), with the properties:

(i.) $\int_G 1 \; d_H g = 1$,

(ii.) For any $h \in G$, $f \in C(G, \mathbb{R}) : \int_G f(hg) d_H g = \int_G f(g) d_H g = \int_G f(gh) d_H g$

(iii.) $\int_G f(g) d_H g \geq 0$ if $f \geq 0$.

The properties of 4.2 say that the integral is normalized, bi-invariant, and linear. From here on, whenever we integrate over any group, it will always be with respect to a normalized bi-invariant Haar integral; we will henceforth omit the "H" in d_H.

Corollary 4.3. There exists a unique normalized bi-invariant $\mathbb{C}$-linear Haar integral $I : C(G, \mathbb{C}) \to \mathbb{C}$ with the same properties (i.)–(iii.) of Theorem 4.2.

Proof. Extend the real integral by linearity: for $f(g) = f_1(g) + i f_2(g)$ where $f_1, f_2 \in C(G, \mathbb{R})$, we have that $\int_G (f_1 + i f_2)(g) dg = \int_G f_1(g) dg + i \int_G f_2(g) dg$ □

Example 4.4. If G is finite with discrete topology (hence compact), then

$$\int_G f(g) dg = \frac{1}{|G|} \sum_{g \in G} f(g).$$

Example 4.5. If $G = S^1 = \{z = e^{i\theta} : \theta \in \mathbb{R}\}$, then

$$\int_G f(g) dg = \int_{-\pi}^{\pi} f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

For more on the normalized invariant Haar integral, see [Fo], chapter 11.1.

4.2 Representations

Definition 4.6. Let V be a real or complex vector space and let G be a Lie group. A *representation* ρ of G on V is a differentiable homomorphism from G into the group of invertible linear transformations of V, i.e., $\rho : G \to GL(V)$.

Remark 4.7. In the definition above, $GL(V)$ is Lie group consisting of all real (resp., complex) linear automorphisms of the real (resp., complex) vector space V.

We will often use the notation ρ_g in place of $\rho(g)$. It can also be convenient to omit specifically mentioning the map ρ and writing simply gv for $\rho_g v$ whenever $v \in V$. In this case, the homomorphism property of ρ reads:

$$(gh)v = g(hv).$$

We may also think of representations of G as linear group (left) *actions* on V.

One often calls the underlying vector space V of a representation a *G-space.* By a further abuse of the nomenclature, one often calls V itself a representation of G.

A *matrix representation* $\varrho : G \to GL(n, \mathbb{F})$ is obtained from a representation by choosing a basis. A matrix representation takes a group to the general linear group of $n \times n$ invertible matrices over a field $\mathbb{F}$.

Definition 4.8. Suppose a group G acts on a complex vector space V. An inner product on V is said to be *G-invariant* if $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$ and $v, w \in V$. A *unitary representation* is a representation together with a G-invariant inner product.

Theorem 4.9. Complex representations of a compact group G are unitary, i.e., they possess a G-invariant inner product.

Proof. Take any Hermitian inner product $H : V \times V \to \mathbb{C}$. For all $u, v \in V$, let

$$\tilde{H}(u, v) = \int_G H(gu, gv) dg.$$

Since $H(gu, gv)$ is a continuous function, it is well-defined. Now, for all $h \in G$, $\tilde{H}(hu, hv) = \int_G H(hgu, hgv)dg = \int_G H(gu, gv)dg = \tilde{H}(u, v)$ by left-invariance of the integral, hence $\tilde{H}(u, v)$ is G-invariant. $\tilde{H}(u, v)$ is also positive definite by the third property of Theorem 4.2. □

Definitions 4.10. A subspace W of the representation V which is invariant under G, i.e., $w \in W \implies gw \in W$, is called a *subrepresentation*.

A nonzero representation W is said to be *irreducible* if there are no proper subrepresentations of W.

Definition 4.11. A *morphism* $f : V \to W$ is a linear map such that $f(gv) = gf(v)$ for all $g \in G$ and $v \in V$. The set of all such morphisms is denoted $\mathrm{Hom}_G(V, W)$.

Theorem 4.12. (Schur's Lemma) Let V and W be irreducible representations of a group G. Then if ϕ is a morphism from V to W

(i.) ϕ is either an isomorphism or the 0 map, and

(ii.) if $V = W$ is a *complex* representation, then $\phi = \lambda I_V$ for some $\lambda \in \mathbb{C}$.

Proof. First we will show that $v \in \ker(\phi)$ implies $gv \in \ker(\phi)$, i.e., the kernel $\ker(\phi)$ is a subrepresentation of V. If $v \in \ker(\phi)$, then $g\phi(v) = g0 = 0$. But since ϕ is a morphism, $\phi(gv) = g\phi(v) = 0$. Hence gv is in $\ker(\phi)$ and so $\ker(\phi)$ is a subrepresentation of V. But since V is irreducible, $\ker(\phi)$ must be either $\{0\}$ or V.

If $\ker(\phi) = V$, then $\ker(\phi)$ is the 0 function.

Suppose $\ker(\phi) = \{0\}$. Then ϕ is injective. Hence ϕ is an isomorphism from V to $\mathrm{Im}(\phi)$, thus $\mathrm{Im}(\phi)$ is a G-space. But $\mathrm{Im}(\phi) \subset W$ and W is irreducible, hence $\mathrm{Im}(\phi) = W$ and V must be isomorphic to W. This proves (i.).

To prove (ii.) let λ be some eigenvalue for ϕ so that $\phi(w) = \lambda w$ for some eigenvector $w \in V$. Now $\phi - \lambda I$ is also a morphism since for all $v \in V$, $g(\phi - \lambda I)(v) = g\phi(v) - \lambda gv = \phi(gv) - \lambda gv = (\phi - \lambda I)(gv)$. But $\ker(\phi - \lambda I) = \{v \in V : (\phi - \lambda I)(v) = 0\} = \{v \in V : \phi(v) = \lambda v\}$, and so $\ker(\phi - \lambda I)$ contains the eigenvector w. Hence $\ker(\phi - \lambda I) \neq \{0\}$ and therefore the morphism $\phi - \lambda I$ is not an isomorphism. So by (i.) of Schur's lemma, $\phi - \lambda I = 0$, i.e., $\phi = \lambda I$.

□

Remark: Part (ii.) above says that $\mathrm{End}_G(V)$ is a complex one-dimensional vector space if V is a complex irreducible representation. On the other hand, if V is an irreducible real representation, then $\mathrm{End}_G(V)$ may be one- or two-dimensional (either isomorphic to $\mathbb{R}$ or to the two-dimensional real algebra $\mathbb{C}$).

Theorem 4.13. If G is compact and abelian, the complex irreducible representations of G are one-dimensional.

Proof. Let V be an irreducible representation of the abelian compact group G. Since G is abelian, $\rho_g(\rho_h(v)) = \rho_{gh}(v) = \rho_{hg}(v) = \rho_h(\rho_g(v))$, hence ρ_g is a morphism. Then by Schur's lemma (4.12), the morphism ρ_g acts as a multiple of the identity. Therefore any subspace of V is also a subrepresentation (because a subspace is closed under scalar multiplication). But since ρ_g acts as scalars and V is irreducible, V must be one-dimensional. □

It is especially convenient that these groups are compact, since we have the following theorem:

Theorem 4.14. (Complete Reducibility Theorem) For a compact group G, any finite-dimensional representation V can be decomposed into the direct sum of irreducible representations, i.e., $V = \bigoplus_{i=1}^{k} I_i^{\mu_i}$, where the I_i's are irreducible representations and μ_i's are the multiplicities.

Proof. Since V is compact, it is unitary (4.9), hence V has a G-invariant inner product $\langle \cdot, \cdot \rangle$. If V is irreducible, the proof is trivial.

Suppose V is reducible. Then there exists a proper subrepresentation $W \subset V$ such that direct summing with its orthogonal complement $W' = \{u \in V : \langle u, w \rangle = 0 \text{ for all } w \in W\}$ is all of V, i.e., $V = W \oplus W'$. We wish to show that W' is also a subrepresentation of V. Let $u \in W'$ and suppose $gu \in W'$. Then for all $w \in W$, $\langle gu, w \rangle = \langle u, g^{-1}w \rangle = 0$ by definition of W' (since $g^{-1}w \in W$). Hence gu is in W' and W' is G-invariant, i.e., a subrepresentation.

The rest of the proof follows by induction on the dimension of V. $\square$

Definition 4.15. The $I_i^{\mu_i}$ in theorem 4.14 is called an *isotypic space*.

Definitions 4.16. The *trivial representation* is a one-dimensional vector space on which G acts as the identity.

The *defining representation* of a group of matrices is the natural action (matrix multiplication) of those matrices on the vector space.

Definition 4.17. If V is a vector space over $\mathbb{C}$, define $\overline{V}$ to be the vector space over $\mathbb{C}$ with the same underlying set as V and with the same additive structure as V, but with conjugate scalar multiplication $\lambda \cdot v = \overline{\lambda} v$ where $\cdot$ denotes the scalar multiplication in $\overline{V}$.

Definition 4.18. If V is a G-space with g acting by ρ_g, then the *conjugate representation* of V, denoted $\overline{V}$, is also a G-space where g acts by $\overline{\rho}_g$.

Definition 4.19. The *dual representation* V^* of a finite-dimensional vector space V over $\mathbb{C}$ is defined as the set of homomorphisms from V to $\mathbb{C}$, i.e., $V^* = \mathrm{Hom}(V, \mathbb{C})$. If g acts by ρ_g on V, then g acts by $\rho_{g^{-1}}^T$ on V^*.

Theorem 4.20. For any unitary representation V of G, we have

$$\overline{V} \simeq V^*.$$

Proof. This follows immediately from the fact that a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V defines a linear isomorphism $i : \overline{V} \to V^*$ by $v \mapsto \langle v, \cdot \rangle$. The fact that i is an isomorphism of representations follows easily from the G-invariance of $\langle \cdot, \cdot \rangle$. $\square$

Definitions 4.21. Given two representations V and W, we can construct a new representation by taking their direct sum $V \oplus W$. Then $V \oplus W$ has the action $g(v \oplus w) = gv \oplus gw$.

New representations can also be made by taking tensor products $V \otimes W$ with the action $g(v \otimes w) = gv \otimes gw$.

Similarly, the symmetric powers $\mathrm{Sym}^k V$ (respectively, the alternating powers $\Lambda^k V$) form a subrepresentation of $V^{\otimes k}$ with the action $g \cdot (v_1 \odot \cdots \odot v_k) = gv_1 \odot \cdots \odot gv_k$ (respectively, $g \cdot (v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k$) extended by linearity.

A good reference to tensor products, exterior products, and symmetric products can be found in appendices B.1 and B.2 of [FuHa].

Example 4.22. If we have two matrix representations $A : G \to GL(m, \mathbb{C})$, $g \mapsto A(g)$, and $B : G \to GL(n, \mathbb{C})$, $g \mapsto B(g)$, then the direct sum representation $A \oplus B : G \to GL(m + n, \mathbb{C})$ is formed by forming block matrices:

$$g \mapsto \begin{pmatrix} A(g) & \\ & B(g) \end{pmatrix}.$$

Example 4.23. The tensor product of two matrix representations A and B is given by the "Kronecker product" of the respective matrices:

$$g \mapsto A(g) \otimes B(g) = \begin{pmatrix} a_{11}B(g) & a_{1m}B(g) & \cdots & a_{1m}B(g) \\ a_{21}B(g) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B(g) & \cdots & \cdots & a_{mm}B(g) \end{pmatrix},$$

where the a_{ij}'s are the entries of $A(g)$.

Definition 4.24. Let $g \in G$, and $h = \exp(tX) \in G$. Then differentiating the conjugation map $C_g : h \mapsto ghg^{-1}$ at $t = 0$ defines the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, or equivalently $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, where $\mathrm{Ad}(g)(X) = \frac{d}{dt}[g(\exp(tX))g^{-1}]|_{t=0}$. In the case of matrix groups, $\mathrm{Ad}(g)(X) = gXg^{-1}$.

Proposition 4.25. If $G \subset \mathrm{GL}(n, \mathbb{C})$ is a matrix group, then its Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ has the natural inner product $\langle X, Y\rangle = \mathrm{Tr}(X^*Y)$. This inner product is Ad-invariant, so it unitarizes the adjoint representation.

Proof. Let $g \in G$. Then

$$\begin{aligned} \langle \mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y)\rangle &= \mathrm{Tr}(\mathrm{Ad}(g)(X)^*, \mathrm{Ad}(g)(Y)) = \mathrm{Tr}((gXg^{-1})^* gYg^{-1}) \\ &= \mathrm{Tr}((g^*)^{-1}X^*g^*gYg^{-1}) = \mathrm{Tr}((g^*)^{-1}g^*X^*Ygg^{-1}) = \mathrm{Tr}(X^*Y) \\ &= \langle X, Y\rangle. \end{aligned}$$

□

4.3 Characters

Representations hold a large amount of information. For instance, there is a matrix, i.e., a linear transformation, attached to each element in a possibly infinite group. But just by studying a special kind of function called the *character* of a representation (which is a scalar valued function!), we can capture that entire representation up to isomorphism without losing any essential information (the proof of this fact can be found at 4.33).

Definition 4.26. Let V be a finite vector space over $\mathbb{C}$ and G be a Lie group. Let $\rho : G \to GL(V)$ be a representation. The *character* $\chi_V : G \to \mathbb{C}$ of a vector space V over $\mathbb{C}$ is given by $\chi_V(g) = \mathrm{Tr}(\rho_g)$.

Properties 4.27.

(i.) Characters are (infinitely) differentiable functions on the group G.

(ii.) If V and W are isomorphic representations, then $\chi_V = \chi_W$.

(iii.) Characters are class functions, i.e., $\chi_V(ghg^{-1}) = \chi_V(h)$.

(iv.) $\chi_{V \oplus W} = \chi_V + \chi_W$.

(v.) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.

(vi.) $\chi_V(e) = \dim(V)$.

(vii.) $\chi_{\overline{V}} = \overline{\chi_V}$.

(viii.) $\chi_{V^*}(g) = \chi_V(g^{-1})$.

(ix.) For compact G, $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)} = \chi_{\overline{V}}$.

Proof. Because ρ_g is a differentiable function, $\mathrm{Tr}(\rho_g)$ is also differentiable, which proves (i.).

For the proof of (ii.), let $f : V \to W$ be an isomorphism, where $\rho_g : V \to V$ and

$\sigma_g : W \to W$. Then $f \circ \rho_g = \sigma_g \circ f$, or equivalently, $\rho_g = f^{-1} \circ \sigma_g \circ f$. Hence $\chi_V = \mathrm{Tr}(\rho_g) = Tr(f^{-1} \circ \sigma_g \circ f) = \mathrm{Tr}(\sigma_g) = \chi_W$.

The proofs for (iii.) to (v.) come directly from properties of the trace.

To prove (vi.), note that $\chi_V(e) = \mathrm{Tr}(\rho_e) = \mathrm{Tr}(\mathrm{Id}_V) = \dim(V)$, where Id_V is the identity matrix on V.

For (vii.), $\chi_{\overline{V}} = \mathrm{Tr}(\overline{\rho_g}) = \overline{\mathrm{Tr}(\rho_g)} = \overline{\chi_V}$.

For (viii.), $\chi_{V^*}(g) = \mathrm{Tr}(\rho^T_{g^{-1}}) = \mathrm{Tr}(\rho_{g^{-1}}) = \chi_V(g^{-1})$.

For (ix.), recall from 4.9 that for a compact G, complex representations are unitary (with a G-invariant inner product). Let $\mathscr{B}$ be an orthonormal basis of V with respect to a G-invariant inner product. Then $[\rho_g]_{\mathscr{B}} \in \mathrm{U}(n)$. But $[\rho_g^{-1}]_{\mathscr{B}} = [(\rho_g)^{-1}]_{\mathscr{B}} = [\rho_g]_{\mathscr{B}}^{-1} = [\rho_g]_{\mathscr{B}}^*$. So by taking the trace, we have $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ (transposing doesn't affect the trace).

□

Definition 4.28. The character of an irreducible representation is called an *irreducible character.*

For compact groups, there is an (Hermitian) inner product on the characters given by

$$\langle \chi_V, \chi_W \rangle = \int_G \overline{\chi_V(g)} \chi_W(g) dg. \tag{4.1}$$

We will compute these types of integrals explicitly in the last chapter.

Proposition 4.29. Let V_0 be the trivial isotypic subspace of a representation V of a group G. Then

$$\int_G \chi_V(g) dg = \dim(V_0)$$

Proof. See [BrDi], chapter 2 (theorem 4.11). □

Proposition 4.30. Let G be a compact group and V, W complex representations of G. Then $\langle \chi_V, \chi_W \rangle = \dim(\mathrm{Hom}_G(V, W))$.

Proof. $\langle \chi_V, \chi_W \rangle = \int_G \overline{\chi_V(g)} \chi_W(g) dg$ (by definition)

$= \int_G \chi_{V^*}(g) \chi_W(g) dg$ (by 4.27(ix) and 4.27(viii))

$= \int_G \chi_{V^* \otimes W}(g) dg$ (by 4.27(v))

$= \int_G \chi_{\mathrm{Hom}(V,W)}(g) dg$ (see [HoKu])

$= \dim(\mathrm{Hom}(V, W)_0)$ (by 4.29),

where $\mathrm{Hom}(V, W)_0$ is the trivial isotypic space of $\mathrm{Hom}(V, W)$. However, it is easy to see that $\mathrm{Hom}(V, W)_0 = \mathrm{Hom}_G(V, W)$: Let $f \in \mathrm{Hom}(V, W)$. On, $\mathrm{Hom}(V, W)_0$, the action of $g \in G$ is trivial: $g \cdot f = f$. On the other hand, the action of G on $Hom(V, W)$ is $g \cdot f = gfg^{-1}$, hence $gf = fg$. So f is a morphism of the G-spaces V and W, i.e., $f \in \mathrm{Hom}_G(V, W)$.

□

We now introduce a very important theorem in the context of compact complex irreducible representations known as the *orthogonality relations for characters.*

Theorem 4.31. (Orthogonality Relations for Characters) If V and W are complex irreducible representations, then

$$\int_G \overline{\chi_V(g)} \chi_W(g) dg = \begin{cases} 1, & \text{if } V \text{ is isomorphic to } W \\ 0, & \text{if } V \text{ is not isomorphic to } W. \end{cases} \tag{4.2}$$

Proof. By 4.30, $\int_G \overline{\chi_V(g)} \chi_W(g) dg = \dim(\mathrm{Hom}_G(V, W))$. If V and W are not isomorphic, then by Schur's lemma (4.12) any morphism from V to W must be 0. Hence $\dim(\mathrm{Hom}_G(V, W)) =$

0. If V and W are isomorphic, then again by Schur's Lemma, $\dim(\mathrm{Hom}_G(V, W)) = 1$ (cf., our Remark after 4.12.) $\square$

Proposition 4.32. The characters of irreducible representations are linearly independent (on the vector space of functions).

Proof. It suffices to show the irreducible characters form an orthogonal set of nonzero functions. Since irreducible representations have dimension 1 or more, their characters cannot be zero. By the orthogonality relations for characters (4.31), characters of irreducible representations are orthogonal. $\square$

Theorem 4.33. A representation is determined by its character up to isomorphism.

Proof. Let V and W be representations of G. Then by complete reducibility (4.14), $V = \bigoplus_i I_i^{m_i}$ and $W = \bigoplus_i I_i^{n_i}$, where the I_i's are irreducible representations. Taking characters, $\chi_V = \sum_i m_i \chi_i$ and $\chi_W = \sum_i n_i \chi_i$. Now, suppose $\chi_V = \chi_W$. Then $\sum_i (m_i - n_i)\chi_i = 0$. But by proposition 4.32, the irreducible characters are linearly independent. Hence $m_i - n_i = 0$, i.e., $m_i = n_i$. So V and W are in fact isomorphic. $\square$

Chapter 5

Representations of Tori

5.1 Representations of Tori

Definition 5.1. If V is a vector space over $\mathbb{R}$ with dimension n, then a *lattice* L is any set $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r$, where $v_1, \cdots, v_r$ are linearly independent vectors in V and r is the rank of L.

L is called a *full lattice* if $r = n$.

Definition 5.2. The *weight lattice* L^* of a full lattice L is a subset of V^* (the dual vector space of V) defined by $L^* = \{f \in V^* : f(L) \subset \mathbb{Z}\}$.

Definition 5.3. A k-torus T^k is any Lie group which is isomorphic to $S^1 \times S^1 \times \cdots \times S^1$ (with k factors).

Then T^k is abelian and compact, being isomorphic to a direct product of abelian compact groups.

Proposition 5.4. The exponential map induces an isomorphism from the (additive) group $\mathfrak{t}/I$ to T, where $I = \ker(\exp)$. I is a full sublattice, i.e., a set of integral combinations of elements in a basis of the vector space $\mathfrak{t}$.

Proof. Since T is abelian (hence $\mathfrak{t}$ is abelian with respect to the Lie bracket), $\exp(X+Y) = \exp(X)\exp(Y)$ for $X, Y \in \mathfrak{t}$. Hence exp is a homomorphism from the additive Lie group $\mathfrak{t}$ to the multiplicative Lie group T. Now, $\exp(\mathfrak{t}) \subset T$. Since the dimension of $\mathfrak{t}$ and T are the same, and since T is connected, $\exp(\mathfrak{t})$ is connected. It follows that $\exp(\mathfrak{t}) = T$ and exp is surjective. Thus, by the First Isomorphism Theorem, $T = \exp(\mathfrak{t})$ is isomorphic to $\mathfrak{t}/\ker(\exp) = \mathfrak{t}/I$. $\square$

We may construct an isomorphism from the Lie algebra of S^1, $\mathrm{Lie}(S^1)$, to $\mathbb{R}$ by identifying $x \in \mathbb{R}$ with $\frac{d}{dt}e^{2\pi ixt}|_{t=0} = 2\pi ix \in \mathrm{Lie}(S^1)$. Under this identification and proposition 5.4, S^1 is isomorphic to $\mathbb{R}/\mathbb{Z}$.

Let $\tilde{\chi} : \mathfrak{t}/I \to \mathbb{R}/\mathbb{Z}$. Then taking the differential of $\tilde{\chi}$ (or equivalently the differential of χ) at the identity gives $\mathrm{D}\chi_e = \mathrm{D}\,\tilde{\chi}_0 : \mathfrak{t} \to \mathbb{R}$.

Letting $\pi_{\mathfrak{t}}$ be the mapping from $\mathfrak{t}$ to $\mathfrak{t}/I$ and $\pi_{\mathbb{R}}$ be the mapping from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$, we have the following commutative diagram:

$$\begin{array}{rccc}
\chi: & T & \longrightarrow & S^1 \\
& \uparrow & & \uparrow \\
\tilde{\chi}: & \mathfrak{t}/I & \longrightarrow & \mathbb{R}/\mathbb{Z} \\
& \uparrow & & \uparrow \\
\mathrm{D}\chi_e: & \mathfrak{t} & \longrightarrow & \mathbb{R}
\end{array}$$

which is a consequence of property (ii.) of the exponential map (Proposition 3.29) and of standard homomorphism theorems.

Proposition 5.5. The correspondence $\chi \mapsto \omega_\chi = \mathrm{D}\chi_e$ is an isomorphism between irreducible characters of T and the weight lattice of T.

Proof. We will first show that ω_χ is in the weight lattice, i.e., $\omega_\chi(I)$ is a subset of $\mathbb{Z}$.

By commutativity of the diagram above, $\pi_{\mathbb{R}}(\omega_\chi(I)) = \tilde{\chi}(\pi_{\mathfrak{t}}(I))$. But every element of I in $\mathfrak{t}$ goes to the 0 element in $\mathfrak{t}/I$, i.e., $0 + I$. Hence $\tilde{\chi}(\pi_{\mathfrak{t}}(I)) = \tilde{\chi}(0 + I)$. Since $\tilde{\chi}$ is a homomorphism (taking identity to identity), we have that $\tilde{\chi}(0 + I) = 0 + \mathbb{Z}$. The only elements in $\mathbb{R}$ that maps to the trivial element in $\mathbb{R}/\mathbb{Z}$ by $\pi_{\mathbb{R}}$ is $\mathbb{Z}$. So $\omega_\chi(I) \subset \mathbb{Z}$.

Next we show that ω_χ is a homomorphism. By the chain rule, $\omega_{\chi\psi} = \mathrm{D}(\chi\psi)_e = \mathrm{D}\,\chi_e\psi(e) + \chi(e)\,\mathrm{D}\,\psi_e = \mathrm{D}\,\chi_e \cdot 1 + 1 \cdot \mathrm{D}\,\psi_e = \omega_\chi + \omega_\psi$.

Now we show injectivity by showing that the only way the weight of a character can be 0 is if the character is trivial. Suppose $\omega_\chi = 0$, then $0 = \pi_{\mathbb{R}} \circ \omega_\chi = \tilde{\chi} \circ \pi_{\mathfrak{t}}$. Hence $\mathrm{Im}(\pi_{\mathfrak{t}}) \subset \ker \tilde{\chi}$. But $\mathrm{Im}(\pi_{\mathfrak{t}}) = \mathfrak{t}/I$ and $\ker \tilde{\chi} \subset \mathfrak{t}/I$. Hence $\ker(\tilde{\chi}) = \mathfrak{t}/I$, and so $\tilde{\chi} = 0$. Since $\chi \circ \exp = \exp \circ \tilde{\chi} = 1$, $\mathrm{Im}(\exp) \subset \ker(\chi) \subset T$. But $\mathrm{Im}(\exp) = T$, hence $\ker(\chi) = T$, i.e.,

$\chi = 1$.

Finally, we will show surjectivity, i.e., given any weight of the torus, we can always find a character of the torus corresponding to that weight. Suppose $\omega \in I^*$. Then $\omega(I) \subset \mathbb{Z} = \ker \pi_{\mathbb{R}}$ hence $\pi_{\mathbb{R}} \circ \omega(I) = 0$. Therefore $I \subset \ker(\pi_{\mathbb{R}} \circ \omega)$. So there exists $\tilde{\omega} : \mathfrak{t}/I \to \mathbb{R}/\mathbb{Z}$ such that $\tilde{\omega} \circ \pi_{\mathfrak{t}} = \pi_{\mathbb{R}} \circ \omega$. The differential of $\tilde{\omega}$ at the identity e is a linear map $\mathrm{D}\tilde{\omega}_e : \mathfrak{t} \to \mathbb{R}$ and the identity map id gives the following diagram:

$$\begin{array}{rccc} \mathrm{D}\,\tilde{\omega}_e : & \mathfrak{t} & \longrightarrow & \mathbb{R} \\ & id \uparrow & & \uparrow id \\ \omega : & \mathfrak{t} & \longrightarrow & \mathbb{R} \end{array}$$

(cf., property (i.) of the exponential map (Proposition 3.29)) so that $\mathrm{D}\,\tilde{\omega}_e \circ id = id \circ \omega$, i.e., $\mathrm{D}\,\tilde{\omega}_e = \omega$

Now, since the exponential maps exp are diffeomorphisms of Lie groups, there exists χ_ω such that $\chi_\omega \circ \exp = \exp \circ \tilde{\omega}$ and χ_ω is a differentiable homomorphism.

The differential of χ_ω at the identity is $(\mathrm{D}\,\chi_\omega)_e : \mathfrak{t} \to \mathbb{R}$, and so $\omega = \mathrm{D}\,\tilde{\omega}_e = (\mathrm{D}\,\chi_\omega)_e$.

So we have found a character χ_ω of the group which also has the property that the differential of that character is ω.

$\square$

Chapter 6

Maximal Tori

The representation theory of non-abelian groups is in general quite complicated, but just by studying one of the largest of the tori of that group (which is a compact, connected, and abelian subgroup), we can greatly simplify the theory.

6.1 Maximal Tori

Remark 6.1. Throughout the remainder of this thesis, G will always be a compact connected Lie group.

Definition 6.2. Let G be a compact Lie group. A torus T is called a *maximal torus* if $T \subset G$ and there is no other torus $T' \subset G$ such that T is a proper subset of T'.

It is possible for there to be more than one maximal tori in G (and usually there are many). We will later see that all of the maximal tori are conjugate to each other (Corollary 6.11).

Proposition 6.3. Let G be a compact connected matrix Lie group with Lie algebra $\mathfrak{g}$. Let T be a maximal torus of G and $\mathfrak{t}$ its Lie algebra. Then T acts on $\mathfrak{g}$ by the adjoint representation, i.e., by matrix conjugation. G splits into the direct sum of irreducible one-dimensional and two-dimensional T-spaces. Let $d = \dim(G)$ and $r = \dim(T)$. Then there are r one-dimensional T-spaces V_0 which T acts on trivially. The two-dimensional (real) T-spaces V_j's are acted on by T as

$$\begin{pmatrix} \cos 2\pi i\vartheta_j & -\sin 2\pi i\vartheta_j \\ \sin 2\pi i\vartheta_j & \cos 2\pi i\vartheta_j \end{pmatrix},$$

where the ϑ_j's are functions that take $\mathfrak{t}$ to $\mathbb{R}$.

Proof. See construction 4.10 and proposition 4.12 of [Ad]. □

The nonzero $\pm\vartheta_j$'s in proposition 6.3 are called *roots*, and are usually the difference of two angles (because of the adjoint representation acting on the spaces by conjugation). There are $\frac{d-r}{2}$ of the V_j's.

If we pass $\mathfrak{g}$ to the complexification $\mathfrak{g}_{\mathbb{C}}$, the adjoint representation of T splits $\mathfrak{g}_{\mathbb{C}}$ into n one-dimensional T-spaces. There are r trivial spaces and $d - r$ non-trivial one-dimensional T-spaces. In the complexified case, T acts on the nontrivial spaces as $e^{2\pi i\vartheta_j}$. In our examples, we will mainly work with complexified vector spaces unless stated otherwise.

One way to check whether or not a torus is maximal is by the following proposition:

Proposition 6.4. A torus $T \subset G$ is maximal if and only if the trivial isotypic space of $\mathfrak{g}$ (in the Adjoint representation) is equal to the Lie algebra $\mathfrak{t}$ of T.

Proof. Suppose $T \subset T'$. Then $\mathfrak{t} \subset \mathfrak{t}' \subset V_0' \subset V_0$. If $\mathfrak{t} = V_0$, then $\mathfrak{t} = \mathfrak{t}'$ and hence $T = T'$, i.e., T is maximal. The converse of the proof is referred to proposition 4.14 of [Ad]. □

Example 6.5. Consider the group U(2). A maximal torus of U(2) has the form

$$T = \left\{ t = \begin{pmatrix} e^{2\pi i\alpha} & \\ & e^{2\pi i\beta} \end{pmatrix} \right\}$$

where $\alpha,\ \beta \in \mathbb{R}$.

The Lie algebra of T, $\mathfrak{t}$, is given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 2\pi i\alpha & \\ & 2\pi i\beta \end{pmatrix} \right\}.$$

The torus acts on its own Lie algebra trivially (and thus T is maximal), and there are $\dim(T) = 2$ copies of the trivial representation. So there must be two other (complex) one-dimensional irreducible nontrivial representations (which are possibly isomorphic to each other).

The Lie algebra of U(2) is given by $\mathfrak{u}(2) = \{X \in \mathfrak{gl}(2, \mathbb{C}) : X^* + X = 0\}$, i.e., $\mathfrak{u}(2)$ is of the form:

$$\mathfrak{u}(2) = \left\{ X = \begin{pmatrix} ia & z \\ -\bar{z} & ib \end{pmatrix} \right\}$$

where $a,\ b \in \mathbb{R}$, and $z \in \mathbb{C}$.

The adjoint representation $\mathrm{Ad}(T)$ acts on $\mathfrak{u}(2)$ by conjugation, so

$$\mathrm{Ad}(t)(X) = tXt^{-1} = \begin{pmatrix} ia & e^{2\pi i(\alpha-\beta)}z \\ -e^{2\pi i(\beta-\alpha)}\bar{z} & ib \end{pmatrix}$$

Hence the roots of U(2) are $\pm(\alpha - \beta)$.

Example 6.6. Now consider USp(4). A maximal torus of USp(4) has the form

$$T = \left\{ t = \begin{pmatrix} e^{2\pi i\alpha} & & & \\ & e^{2\pi i\beta} & & \\ & & e^{-2\pi i\alpha} & \\ & & & e^{-2\pi i\beta} \end{pmatrix} \right\}$$

where α, $\beta \in \mathbb{R}$.

Its Lie algebra $\mathfrak{t}$, is given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 2\pi i\alpha & & & \\ & 2\pi i\beta & & \\ & & -2\pi i\alpha & \\ & & & -2\pi i\beta \end{pmatrix} \right\}.$$

There are $\dim(T) = 2$ copies of the trivial representation.

The Lie algebra of USp(4) is given by $\mathfrak{usp}(4) = \{X \in \mathfrak{u}(4) : X^T J + JX = 0\}$, i.e., $\mathfrak{usp}(4)$ is of the form:

$$\mathfrak{usp}(4) = \left\{ X = \begin{pmatrix} ix_1 & z & w_1 & w_2 \\ -\bar{z} & ix_2 & w_2 & w_3 \\ -\bar{w}_1 & -\bar{w}_2 & -ix_1 & -z \\ -\bar{w}_2 & -\bar{w}_3 & \bar{z} & -ix_2 \end{pmatrix} \right\}$$

where $x_1, x_2 \in \mathbb{R}$, and $w_1, w_2, w_3 \in \mathbb{C}$.

The adjoint representation $\mathrm{Ad}(T)$ acts on $\mathfrak{usp}(4)$ by conjugation, so

$$\mathrm{Ad}(t)(X) = tXt^{-1} = \begin{pmatrix} ix_1 & e^{2\pi i(\alpha-\beta)}z & e^{2\pi i(2\alpha)}w_1 & e^{2\pi i(\alpha+\beta)}w_2 \\ -e^{-2\pi i(\alpha-\beta)}\bar{z} & ix_2 & e^{2\pi i(\beta+\alpha)}w_2 & e^{2\pi i(2\beta)}w_3 \\ -e^{-2\pi i(2\alpha)}\bar{w}_1 & -e^{-2\pi i(\beta+\alpha)}\bar{w}_2 & -ix_1 & -e^{2\pi i(\beta-\alpha)}z \\ -e^{-2\pi i(\alpha+\beta)}\bar{w}_2 & -e^{-2\pi i(\beta)}\bar{w}_3 & e^{-2\pi i(\beta-\alpha)}\bar{z} & -ix_2 \end{pmatrix}$$

Hence the roots of USp(4) are $\pm(\alpha-\beta)$, $\pm(\alpha+\beta)$, $\pm(2\alpha)$, and $\pm(2\beta)$.

In general,

Proposition 6.7.

For $n \geq 2$:

(i.) The roots for U(n) and SU(n) are $\alpha_\mu - \alpha_\nu$ for $1 \leq \mu, \nu \leq n$ where $\mu \neq \nu$.

(ii.) The roots for SO($2n$) are $\pm\alpha_\mu \pm \alpha_\nu$, for $1 \leq \mu < \nu \leq n$.

(iii.) The roots for SO($2n+1$) are $\pm\alpha_\mu \pm \alpha_\nu$, for $1 \leq \mu < \nu \leq n$ and $\pm\alpha_\mu$ for $1 \leq \mu \leq n$.

(iv.) The roots for USp($2n$) are $\pm\alpha_\mu \pm \alpha_\nu$, for $1 \leq \mu < \nu \leq 2n$ and $\pm 2\alpha_\mu$ for $1 \leq \mu \leq 2n$.

Proof. See chapter 5.6 of [BrDi]. □

Definition 6.8. Let G be a topological group and g be a generator of a subgroup $H \subset G$. Then g is a (topological) *generator* of G if the closure of H is G.

Proposition 6.9. Every torus has a topological generator.

Proof. [Ad], Proposition 4.3. □

The next theorem provides one of the main reasons why one would study the maximal tori of a compact connected Lie group G. The theorem says that a maximal torus's conjugates (other maximal tori) fill the entire group G.

Theorem 6.10. If T is a maximal torus of a compact connected group G, then $g \in G$ is an element of some conjugate of T.

Proof. The proof for this theorem is fairly lengthy and detailed - we refer the reader to [Ad] theorem 4.21. □

Corollary 6.11. Any two maximal tori T, T' in G are conjugate, i.e., $T' = gTg^{-1}$ for some $g \in G$.

Proof. Let T and T' be maximal tori of G and let t' be a generator of T' (Proposition 6.9). By Theorem 6.10, $t' \in gTg^{-1}$ for some $g \in G$. Thus $T' \subset gTg^{-1}$. But since T' is a maximal torus, $T' = gTg^{-1}$. □

It follows that the maximal tori of a group G have the same dimension.

Definition 6.12. We denote by $\mathrm{rank}(G)$ the dimension of any (hence all) maximal tori of the compact connected Lie group G.

6.2 The Weyl Group

Proposition 6.13. Let $C_G(T)$ denote the centralizer of a maximal torus T. Then $C_G(T) = T$.

Proof. Since T is abelian, $T \subset C_G(T)$. To show $C_G(T) \subset T$, see [Ad] proposition 4.5. □

Definition 6.14. The *Weyl group*, W of G is defined as $N_G(T)/C_G(T)$, where $N_G(T)$ is the normalizer of T in G. By proposition 6.13, we can redefine the Weyl group to be $W = N_G(T)/T$.

The conjugation action of G restricted to $N_G(T)$ leads to an action of W on T. This action is faithful (i.e., one to one) by proposition 6.13.

The infinitesimal action of W is obtained by restricting the adjoint action of G to $N_G(T)$. Hence the W acts faithfully on $\mathfrak{t}$ ($\mathfrak{t}$ is a representation of the finite group W).

Now W acts on $\mathfrak{t}^*$, in the following way: if ρ is a representation of W on $\mathfrak{t}$ and ρ^* is the dual representation, then for all $w \in W$ and $f \in \mathfrak{t}^*$,

$$\rho_w^*(f)(v) = f(\rho_w^{-1}(v)). \tag{6.1}$$

In other words, the elements of the Weyl group act as the transpose of the inverse on $\mathfrak{t}^*$

Proposition 6.15. The Weyl group is finite.

Proof. See theorem 1.5 in chapter 4 of [BrDi]. □

Proposition 6.16. W acts on the weights of T, i.e., W sends weights to weights.

Proof. Let $w \in W$. By commutativity, $\exp(w(I)) = w(\exp(I))$, where I is the integer lattice. Letting e be the identity element of T, $I = \exp^{-1}(e)$, hence $\exp(I) = \{e\}$. So we have that $\exp(w(I)) = w(\exp(I)) = w(\{e\}) = \{e\}$, i.e., $w(I) \subset \exp^{-1}(\{e\}) = I$, or equivalently, $w^{-1}(I) \subset I$. Now if ϕ is any weight, then by definition $\phi(I) \subset \mathbb{Z}$. By 6.1, $w(\phi)(I) = \phi(w^{-1}(I)) \subset \phi(I) \subset \mathbb{Z}$. Hence $w(\phi)$ is a weight.

□

Chapter 7

Roots and Weights

7.1 The Stiefel Diagram

Definition 7.1. Let G be a compact connected group with a maximal torus T. For each root ϑ_j, let $U_j = \{t \in T : \vartheta_j(t) \in \mathbb{Z}\} = \ker \vartheta_j$.

Example 7.2. The group $SU(2)$ has a root $\vartheta_1 = 2\alpha$. Thus U_1 contains the elements of T such that $\alpha = 0$ or $\alpha = \frac{n}{2}$ for all $n \in \mathbb{Z}$, i.e.,

$$U_1 = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} e^{\pi i n} & \\ & e^{\pi i n} \end{pmatrix} \right\}$$

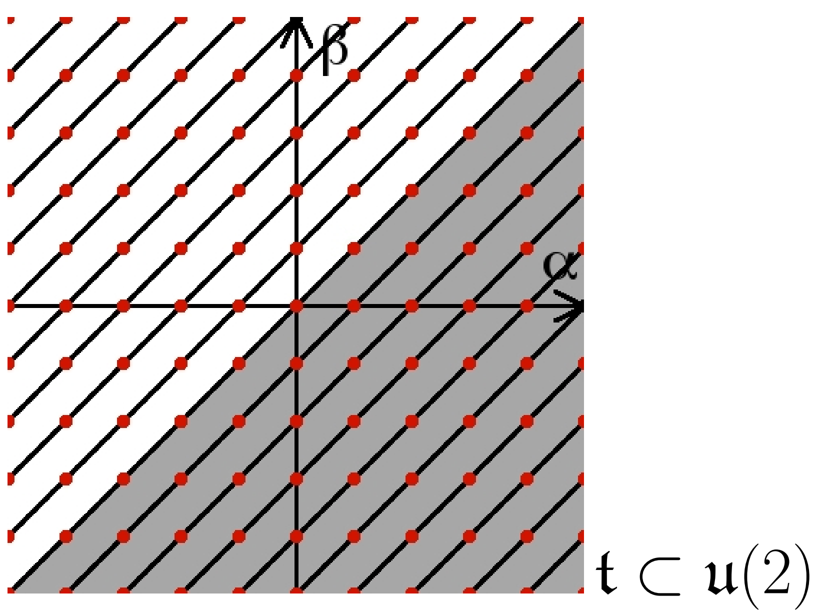
The Lie algebra of U_j, denoted by $L(U_j)$ leads us next to

Definition 7.3. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus of a group with dimension d. The *Stiefel diagram* consists of hyperplanes $L(U_j)$ in $\mathfrak{t}$ for each root ϑ_j. In other words, the Stiefel diagram is made up of spaces of dimension $d-1$ in $\mathfrak{t}$ satisfying $\vartheta_j = 0$ for each root. The *affine Stiefel diagram* consists of all hyperplanes $\vartheta_j = n$, for each $n \in \mathbb{Z}$.

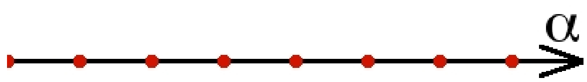
Definitions 7.4. The hyperplanes of the Stiefel diagram divide $\mathfrak{t}$ into open regions called *Weyl chambers*. These regions are the sets of the form $\{t \in \mathfrak{t} : \epsilon_\vartheta \vartheta_j(t) > 0, \text{ for each root } \vartheta_j\}$, where ϵ_ϑ is ± 1.

From this point on, we will fix a Weyl chamber and call it the *fundamental Weyl chamber* FWC. The roots taking positive values in the FWC are called *positive roots*. Out of each pair $\pm\vartheta_j$ of roots, exactly one is positive; we will henceforth assume the roots are chosen so that, for each pair of roots $\pm\vartheta_j$, we have that ϑ_j is positive and $-\vartheta_j$ negative.

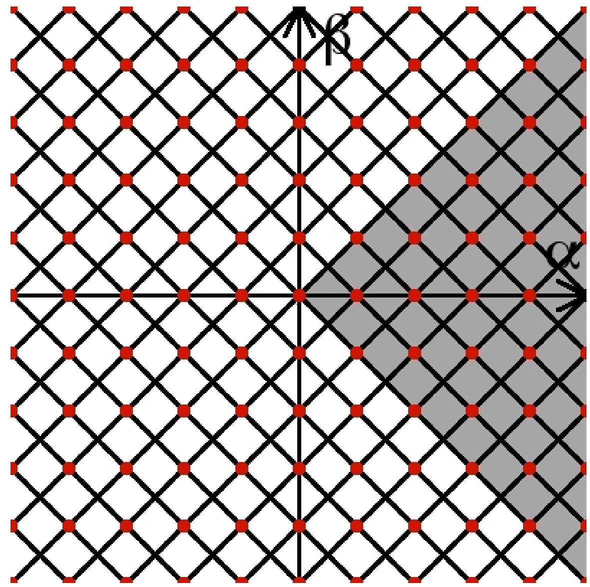
Example 7.5. The Stiefel diagram of U(2) is depicted below. The roots of U(2) are $\pm(\alpha - \beta)$ (see 6.5). Setting the roots equal to an integer constructs the affine Stiefel diagram on the Lie algebra $\mathfrak{t}$ of U(2). The red points in the figure mark the integer lattice and the shaded region is the fundamental Weyl chamber (here we define the FWC by $\alpha - \beta > 0$).



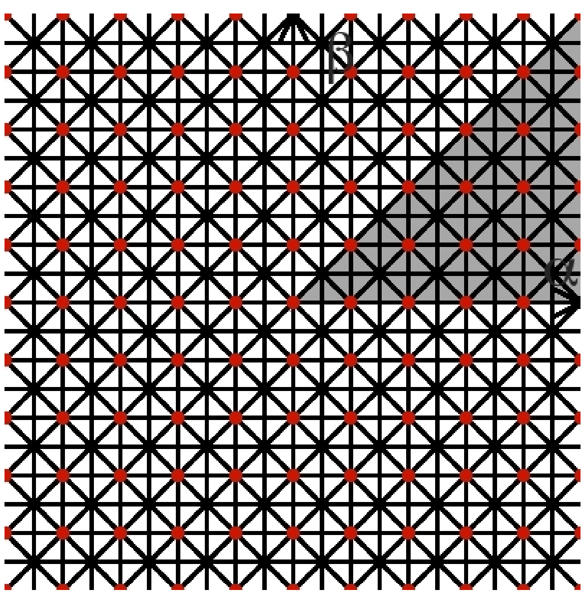
Examples 7.6.



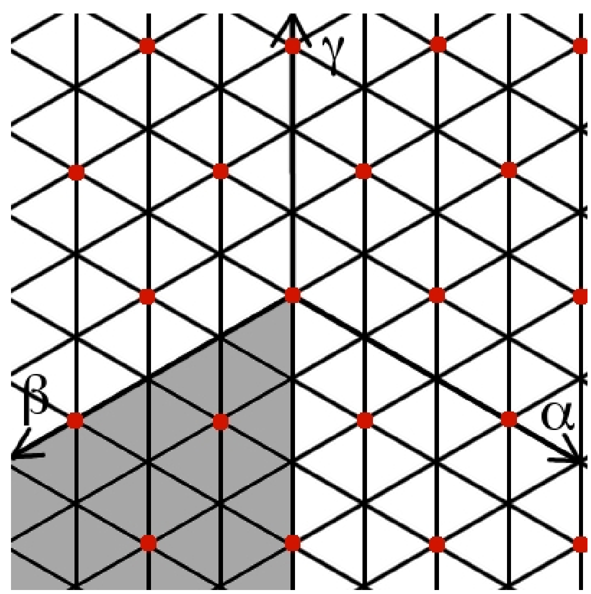
$\mathfrak{t} \subset \mathfrak{u}(1)$



$\mathfrak{t} \subset \mathfrak{so}(4)$



$\mathfrak{t} \subset \mathfrak{usp}(4)$



$\mathfrak{t} \subset \mathfrak{su}(3)$

Proposition 7.7. The Weyl group W is generated by reflections over the hyperplanes.

Proof. See [BrDi], chapter 5, theorem 2.12. □

7.2 The Affine Weyl Group and the Fundamental Group

Definition 7.8. The reflections of all hyperplanes of the affine diagram generate the *affine Weyl group* (or extended Weyl group) Γ.

Γ consists of all the transformations of the Lie algebra of the torus which cover the action of the the Weyl group on the maximal torus. In other words, Γ is the group of operations over the affine diagram that takes the form of those operations on the torus after exponentiation.

Definition 7.9. We define Γ_0 as the intersection of Γ and the group of translations of $\mathfrak{t}$.

The next proposition shows that Γ_0 is a subgroup of the integer lattice I.

Proposition 7.10. If γ_j is the reflection of the origin of $\mathfrak{t}$ by the hyperplane $\vartheta_j = 1$, then Γ_0 is a subgroup of I generated by each γ_j.

Proof. See [Ad], proposition 5.48. □

Example 7.11. Consider the group U(2). The roots are $\pm(\alpha - \beta)$ and the integer lattice is $I = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{Z}\}$. The affine Weyl group Γ contains the identity element and the elements $f_p = p$ for all $p \in \mathbb{Z}$, where f_p's reflect over each hyperplane of $\mathfrak{t}$. Then $\Gamma_0 = \{(\alpha, -\alpha) : \alpha \in \mathbb{Z}\}$.

Some of the major results of our study depend on the premise of a group being simply connected. We will use the fundamental group $\pi_1(G)$ of the Stiefel diagram to determine the simply connectedness of a group using the following theorem:

Theorem 7.12. $\pi_1(G)$ is isomorphic to I/Γ_0.

Proof. We refer the reader to [Ad], theorem 5.47. □

Examples 7.13. For SU(2), the root is $\pm 2\alpha$. The integer lattice is $I = \{\alpha : \alpha \in \mathbb{Z}\} = \mathbb{Z}$. By proposition 7.10, Γ_0 is generated by the reflections over the hyperplanes $\alpha = 1/2$ and $\alpha = -1/2$, which are simply the integers. Hence $\pi_1(\mathrm{SU}(2)) = I/\Gamma_0 = \mathbb{Z}/\mathbb{Z} = 0$.

For U(2), $I = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{Z}\}$. If we define a map $\tau : I \to \mathbb{Z}$, such that $(\alpha, \beta) \mapsto \alpha+\beta$, then I is isomorphic to $\mathbb{Z}^2$. Also, since $\Gamma_0 = \{(\alpha, -\alpha) : \alpha \in \mathbb{Z}\}$ is isomorphic to $\mathbb{Z}$. Thus $\pi_1(\mathrm{U}(2)) = I/\Gamma_0 = \mathbb{Z}^2/\mathbb{Z} = \mathbb{Z}$. Thus, the group U(2) is not simply connected.

For USp(4), the roots are $\pm(\alpha - \beta), \pm(2\alpha), \pm(2\beta), \pm(\alpha + \beta)$. $I = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{Z}\}$, which under the isomorphism τ as introduced above, is isomorphic to $\mathbb{Z}^2$. Γ_0 is generated by the reflections over the hyperplanes $\alpha-\beta = \pm 1$, $\alpha = \pm 1/2$, $\beta = \pm 1/2$, and $\alpha+\beta = \pm 1$ which is identically I. Hence $\pi_1(\mathrm{USp}(4)) = I/\Gamma_0 = \mathbb{Z}^2/\mathbb{Z}^2 = 0$ and USp(4) is simply connected.

In general,

Proposition 7.14.

(i.) SU(n) has the fundamental group $\pi_1(\mathrm{SU}(n)) = 0$.

(ii.) U(n) has the fundamental group $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$.

(iii.) For $n \neq 2$, SO(n) has the fundamental group $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2 = \{0, 1\}$.

In the special case of SO(2), the fundamental group is $\mathbb{Z}$.

(iv.) USp($2n$) has the fundamental group $\pi_1(\mathrm{USp}(2n)) = 0$.

Proof. See 5.49 of [Ad]. □

7.3 Dual Space of the Lie algebra of a Maximal Torus

Choose an Ad_G-invariant inner product on $\mathfrak{g}$. This defines an G-isomorphism $\iota : \mathfrak{g} \to \mathfrak{g}^*$, by $\iota : X \mapsto \langle \cdot , X \rangle$. Similarly, ι identifies the Lie algebra $\mathfrak{t}$ of a maximal torus T of G with its dual algebra $\mathfrak{t}^*$.

Definition 7.15. The *fundamental dual Weyl chamber (FDWC)* is the image $\iota(FWC) \subset \mathfrak{t}^*$ of the FWC under the isomorphism ι.

Example 7.16. Consider U(2). The roots are $\pm(\alpha - \beta)$ (see 6.5). We choose a basis for $\mathfrak{t}$ to be $X = \begin{pmatrix} 2\pi i & \\ & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \\ & 2\pi i \end{pmatrix}$.

Note that the angle between X and Y is $\pi/2$ since $\cos\theta = \frac{\langle X,Y \rangle}{||X||||Y||} = \frac{0}{(2\pi)^2} = 0$.

The elements of $\mathfrak{t}$ are of the form $\alpha X + \beta Y$ and the integer lattice $I = \ker(\exp) = \{a \in \mathfrak{t} : \exp(a) = e\}$ consists of all elements $\alpha X + \beta Y$ of $\mathfrak{t}$ such that $\alpha,\ \beta \in \mathbb{Z}$. Set an inner product on $\mathfrak{t}$ as $\langle A, B \rangle = Tr(B^*A) = Tr(-BA) = -Tr(BA) = -Tr(AB)$, for $A,\ B \in \mathfrak{t}$. Then using the isomorphism ι to identify $\mathfrak{t}$ with its dual space $\mathfrak{t}^*$, we can find a basis X^* and Y^* for $\mathfrak{t}^*$:

$X^*(\alpha X + \beta Y) = \iota(X)(\alpha X + \beta Y) = \langle \alpha X + \beta Y, X \rangle = \alpha \langle X, X \rangle + \beta \langle Y, X \rangle = 4\pi^2\alpha$,

and similarly, $Y^*(\alpha X + \beta Y) = 4\pi^2\beta$.

The weight lattice is then

$$
\begin{aligned}
I^* &= \{f \in \mathfrak{t}^* : f(z) \in \mathbb{Z}, \text{ for all } z \in I\} \\
&= \{f \in \mathfrak{t}^* : f(z) \in \mathbb{Z}, \text{ for all } z = \alpha X + \beta Y, \text{ where } \alpha, \beta \in \mathbb{Z}\} \\
&= \{f \in \mathfrak{t}^* : f(\alpha X + \beta Y) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \mathbb{Z}\} \\
&= \{f \in \mathfrak{t}^* : \alpha f(X) + \beta f(Y) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \mathbb{Z}\} \qquad \text{(since } f \text{ is linear)} \\
&= \{f \in \mathfrak{t}^* : f(X), f(Y) \in \mathbb{Z}\} \\
&= \{\lambda X^* + \mu Y^* : (\lambda X^* + \mu Y^*)(X), (\lambda X^* + \mu Y^*)(Y) \in \mathbb{Z}\} \\
&= \{\lambda X^* + \mu Y^* : (\lambda\langle X, X\rangle + \mu\langle X, Y\rangle) \in \mathbb{Z}, \text{ and } (\lambda\langle Y, X\rangle + \mu\langle Y, Y\rangle) \in \mathbb{Z}\} \\
&= \{\lambda X^* + \mu Y^* : 4\pi^2\lambda = m \in \mathbb{Z}, \text{ and } 4\pi^2\mu = n \in \mathbb{Z}\} \\
&= \{\frac{m}{4\pi^2}X^* + \frac{n}{4\pi^2}Y^* : m, n \in \mathbb{Z}\} \\
&= \{m\alpha + n\beta : m, n \in \mathbb{Z}\}.
\end{aligned}
$$

In general, let V over $\mathbb{R}$ be an n-dimensional vector space with inner product $\langle \cdot\,, \cdot \rangle$. Let L be a lattice so that $L = \mathbb{Z}v_1 + ... + \mathbb{Z}v_n$ for all $v_j \in V$, $1 \le j \le n$. Then the dual space V^* has basis $v_i^* = \iota(v_i)$. If $M = (\langle v_i, v_j\rangle)$ is a matrix of inner products with basis $v_1, \cdots v_n$, then the dual lattice is $L^* = \mathbb{Z}f_1 + ... + \mathbb{Z}f_n$, where

$$
\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = M^{-1} \begin{pmatrix} v_1^* \\ \vdots \\ v_n^* \end{pmatrix}.
$$

Chapter 8

Representation Theory of the Classical Lie Groups

8.1 Representation Theory

As usual, G will be a compact connected Lie group and our vector spaces are finite dimensional.

Theorem 8.1. (Weyl Integration Formula) Let T be a maximal torus of G, W be the Weyl group and $\delta = \prod_{j=1}^{m}(e^{\pi i \vartheta_j(t)} - e^{-\pi i \vartheta_j(t)})$ taken over the m positive roots of G. Then for all class functions f on G,

$$\int_G f(g) = \frac{1}{|W|}\int_T f(t)\delta\bar{\delta}.$$

Proof. The proof can be found in [Ad], theorem 6.1. $\square$

The Weyl Integration Formula gives a simpler way to integrate class functions (such as characters) over an entire group.

The characters of G lead to (Weyl) symmetric characters of T.

Definition 8.2. Let ω be a weight of G. Let $\mathrm{Orb}_W(\omega)$ be the orbit of ω by W. Then the *elementary symmetric sum* $S(\omega)$ is given by

$$S(\omega) = \sum_{\eta \in \mathrm{Orb}_W(\omega)} e^{2\pi i \eta}$$

Example 8.3. Consider the group $\mathrm{U}(2)$. The weights are of the form $m\alpha + n\beta$ and W is a two-element group consisting of the trivial element and the element that sends $m\alpha + n\beta$ to $m\beta + n\alpha$. If a weight of $\mathrm{U}(2)$ is $\omega = m\alpha + n\beta$ and $m \neq n$, then $\mathrm{Orb}_W(\omega) = \{m\alpha + n\beta, m\beta + n\alpha\}$. So $S(\omega) = e^{2\pi i(m\alpha+n\beta)} + e^{2\pi i(m\beta+n\alpha)}$. If $m = n$, then $\mathrm{Orb}_W(\omega) = \{m\alpha + m\beta\}$. So $S(\omega) = e^{2\pi i(m\alpha+m\beta)}$.

We now define a partial ordering of the weights in the weight lattice.

Definition 8.4. Let ω_1 and ω_2 be weights in $\mathfrak{t}^*$. We say that $\omega_1 \leq \omega_2$ if ω_1 is contained in the convex hull of the W-orbit of ω_2. If $\omega_1 < \omega_2$, we say that ω_1 is a *lower* weight than ω_2, and ω_2 is a *higher* weight than ω_1.

This defines a well-defined partial ordering of W-orbits.

Proposition 8.5. If two weights are are both higher and lower than each other, then they must be Weyl group conjugates.

Proof. See property 6.27(iii) of [Ad]. □

Definition 8.6. Let V be a G-space (or a T-space invariant under the Weyl action W). Then the weights of V are invariant under W. Assume V has a weight $\omega_{\max}$ which is higher than all weights ω of V. Then $\omega_{\max}$ is called a *highest weight* of V.

It is an immediate consequence of proposition 8.5 that $\omega_{\max}$ is unique in the closure of the FDWC (if it exists). Sometimes we call this *the* highest weight.

Proposition 8.7. Let χ be a character of a complex irreducible representation. Then we can rewrite χ restricted to a maximal torus T as:

$$\chi|_T = S(\omega) + \text{ lower terms.}$$

where ω is the highest weight in the FDWC.

Proof. See [Ad], proposition 6.33. □

In other words, proposition 8.7 says that a complex irreducible representation has a unique highest weight with multiplicity 1.

Theorem 8.8. (Parameterization of Irreducible Representations by Their Highest Weight) Let G be a compact connected Lie group with a maximal torus T and weight lattice I^*. Let W be the Weyl group, and FDWC denote the fundamental dual Weyl chamber There is a bijection Υ between the set of isomorphism classes of irreducible representations of G and the semi-lattice $I^* \cap FDWC$ such that for any (class of) irreducible representations ρ of G, $\Upsilon(\rho)$ is the unique highest weight of $\rho|_T$ in the FDWC.

Proof. See [Ad], proposition 6.33. □

Proposition 8.9. If ρ is an irreducible representation of G with highest weight $\omega_1 \in$ FDWC and σ is another irreducible representation of G with highest weight $\omega_2 \in$ FDWC, then $\rho \otimes \sigma$ has a unique highest weight $\omega_1 + \omega_2 \in$ FDWC.

Proof. Let V be the G-space corresponding to ρ and W the G-space corresponding to σ. Then by 8.7, for $\rho|_T$, $V = \bigoplus_{\alpha \in \mathrm{Orb}_W(\omega_1)} V_\alpha \oplus$ (lower weight representations).

For $\sigma|_T$, $W = \bigoplus_{\alpha \in \mathrm{Orb}_W(\omega_2)} V_\alpha \oplus$ (lower weight representations).

By taking the tensor product of the characters of V and W, we have (by 4.27(v.) and 8.7) $\chi_{V \otimes W} = \chi_V \cdot \chi_W = (S(\omega_1) + \text{(lower terms)}) \cdot (S(\omega_2) + \text{(lower terms)}) = S(\omega_1 + \omega_2) +$ (lower terms).

Hence for $(\rho \otimes \sigma)|_T$, $V \otimes W = \bigoplus_{\alpha \in \mathrm{Orb}_W(\omega_1+\omega_2)} V_\alpha \oplus$ (lower weight representations).

□

Proposition 8.10. If $\omega \in$ FDWC is any highest weight of the (reducible or not) representation ρ of G, then ρ contains an irreducible subrepresentation with highest weight ω.

Proof. Restrict ρ to a maximal torus and take the space V_ω corresponding to the highest weight ω. Then by theorem 8.8, acting on V_ω with the group G gives a unique (with respect to the Weyl orbit) irreducible subrepresentation with highest weight ω.

□

Theorem 8.11. Let G be a compact connected Lie group with a maximal torus T, I^* the weight lattice of G, and FDWC the fundamental dual Weyl chamber Let $\rho_1, \cdots, \rho_k$ be a finite set of irreducible representations of G with highest weights $\omega_1, \cdots, \omega_k$ that generate the semi-lattice $I^* \cap FDWC$. Then for any weight $\omega \in FDWC$, we have an expression $\omega = n_1\omega_1 + \cdots + n_k\omega_k$, for $n_i \geq 0$. Accordingly, the tensor product $\rho_1^{n_1} \otimes \cdots \otimes \rho_k^{n_k}$ contains a unique irreducible subrepresentation with highest weight ω.

Proof. From propositions 8.9 and 8.10, it follows that the tensor product $\rho_1^{n_1} \otimes \cdots \otimes \rho_k^{n_k}$ contains a unique irreducible subrepresentation with highest weight ω. □

Proposition 8.12. If G is simply connected, then the minimum such k is $k = \mathrm{rank}(G)$ in theorem 8.11, and the corresponding weights $\omega_1, \cdots, \omega_k$ are a basis of the semi-lattice $I^* \cap FDWC$.

Proof. See theorem 5.62 of [Ad]. □

Example 8.13. The group U(2) has roots $\pm(\alpha - \beta)$ and has a weight lattice $I^* = \{p\alpha + q\beta : p, q \in \mathbb{Z}\}$. The Weyl group has two elements (namely, the identity element and the reflection element). The trivial representation is V_0. The defining representation is $V_1 = \mathbb{C}^2$, so that $v \mapsto g \cdot v = gv$ with character (restricted to the maximal torus) $\chi_{V_1} = e^{2\pi i\alpha} + e^{2\pi i\beta}$ and highest weights α and β.

To find the next highest weight representation, take the tensor product of V_1 with itself so that

$$\begin{pmatrix} e^{2\pi i\alpha} & \\ & e^{2\pi i\beta} \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi i(2\alpha)} & & & \\ & e^{2\pi i(\alpha+\beta)} & & \\ & & e^{2\pi i(\alpha+\beta)} & \\ & & & e^{2\pi i(2\beta)} \end{pmatrix}$$

The character of $V_1 \otimes V_1$ is $\chi_{V_1 \otimes V_1} = e^{2\pi i(2\alpha)} + 2e^{2\pi i(\alpha+\beta)} + e^{2\pi i(2\beta)}$.

From linear algebra, the tensor product of a vector space with itself is the direct product of the symmetric space and the alternating space, i.e., $V_1 \otimes V_1 = \mathrm{Sym}^2 V_1 \oplus \Lambda^2 V_1$. Consider $\Lambda^2 V_1$. If V_1 has basis $\{e_1, e_2\}$, then a basis for $\Lambda^2 V_1$ is $e_1 \otimes e_2 - e_2 \otimes e_1$.

For $g \in \mathrm{U}(2)$, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then

$$\begin{aligned}
g \cdot (e_1 \otimes e_2 - e_2 \otimes e_1) &= ge_1 \otimes ge_2 - ge_2 \otimes ge_1 \\
&= (ae_1 + ce_2) \otimes (be_1 + de_2) - (be_1 + de_2) \otimes (ae_1 + ce_2) \\
&= ae_1 \otimes be_1 + ce_2 \otimes be_1 + ae_1 \otimes de_2 + ce_2 \otimes de_2 \\
&\quad - be_1 \otimes ae_1 - de_2 \otimes ae_1 - be_1 \otimes ce_2 - de_2 \otimes ce_2 \\
&= bc(e_2 \otimes e_1) + ad(e_1 \otimes e_2) - ad(e_2 \otimes e_1) - bc(e_1 \otimes e_2) \\
&= (ad - bc)(e_1 \otimes e_2 - e_2 \otimes e_1) \\
&= \det(g)(e_1 \otimes e_2 - e_2 \otimes e_1)
\end{aligned}$$

Thus $\Lambda^2 V_1$ is the determinant representation with character $\chi_{\Lambda^2 V_1} = e^{2\pi i(\alpha+\beta)}$ and highest weight $\alpha + \beta$. Hence $\chi_{V_1 \otimes V_1} = \chi_{\mathrm{Sym}^2 V_1} \oplus \chi_{\Lambda^2 V_1} = (e^{2\pi i(2\alpha)} + e^{2\pi i(\alpha+\beta)} + e^{2\pi i(2\beta)}) + (e^{2\pi i(\alpha+\beta)})$, which implies that $\chi_{\mathrm{Sym}^2 V_1} = e^{2\pi i(2\alpha)} + e^{2\pi i(\alpha+\beta)} + e^{2\pi i(2\beta)} = \frac{e^{2\pi i(3\alpha)} - e^{2\pi i(3\beta)}}{e^{2\pi i(\alpha)} - e^{2\pi i(\beta)}}$.

We will now show that $\chi_{\mathrm{Sym}^2 V_1}$ is irreducible using the Weyl integration formula and the orthogonality relations for characters:

$$\int_{\mathrm{U}(2)} \chi_{\mathrm{Sym}^2_{V_1}} \overline{\chi_{\mathrm{Sym}^2_{V_1}}} dg$$
$$= \frac{1}{2}\int_0^1\int_0^1 \frac{e^{2\pi i(3\alpha)} - e^{2\pi i(3\beta)}}{e^{2\pi i(\alpha)} - e^{2\pi i(\beta)}} \frac{e^{-2\pi i(3\alpha)} - e^{-2\pi i(3\beta)}}{e^{-2\pi i(\alpha)} - e^{-2\pi i(\beta)}} (e^{\pi i(\alpha-\beta)} - e^{-\pi i(\alpha-\beta)})(e^{-\pi i(\alpha-\beta)} - e^{\pi i(\alpha-\beta)}) d\alpha d\beta$$
$$= \frac{1}{2}\int_0^1\int_0^1 \frac{e^{2\pi i(3\alpha)} - e^{2\pi i(3\beta)}}{e^{\pi i(\alpha+\beta)}\left(e^{\pi i(\alpha-\beta)} - e^{\pi i(\beta-\alpha)}\right)} \frac{e^{-2\pi i(3\alpha)} - e^{-2\pi i(3\beta)}}{e^{-\pi i(\alpha+\beta)}\left(e^{-\pi i(\alpha-\beta)} - e^{-\pi i(\beta-\alpha)}\right)} \times$$
$$(e^{\pi i(\alpha-\beta)} - e^{-\pi i(\alpha-\beta)})(e^{-\pi i(\alpha-\beta)} - e^{\pi i(\alpha-\beta)}) d\alpha d\beta$$
$$= \frac{1}{2}\int_0^1\int_0^1 \frac{e^{2\pi i(3\alpha)} - e^{2\pi i(3\beta)}}{e^{\pi i(\alpha+\beta)}} \frac{e^{-2\pi i(3\alpha)} - e^{-2\pi i(3\beta)}}{e^{-\pi i(\alpha+\beta)}} d\alpha d\beta$$
$$= \frac{1}{2}\int_0^1\int_0^1 (e^{2\pi i(3\alpha)} - e^{2\pi i(3\beta)})(e^{-2\pi i(3\alpha)} - e^{-2\pi i(3\beta)}) d\alpha d\beta$$
$$= \frac{1}{2}\int_0^1\int_0^1 2 - e^{2\pi i(3(\alpha-\beta))} - e^{2\pi i(3(\beta-\alpha))} d\alpha d\beta$$
$$= \frac{1}{2}(2+0) = 1$$

Thus, $\chi_{\mathrm{Sym}^2 V_1}$ is irreducible.

Notice that the degree of the symmetric tensor power did not affect the output of integral. Therefore, we may conclude that any of the symmetric tensor powers $\chi_{\mathrm{Sym}^k V_1} = e^{2\pi i(k\alpha)} + e^{2\pi i((k-1)\alpha+\beta)} + \cdots + e^{2\pi i(\alpha+(k-1)\beta)} + e^{2\pi i(k\beta)} = \frac{e^{2\pi i((k+1)\alpha)} - e^{2\pi i((k+1)\beta)}}{e^{2\pi i(\alpha)} - e^{2\pi i(\beta)}}$ are irreducible for all $k \in \mathbb{N}$.

So far we have identified the weights $0, k\alpha, k\beta,$ and $\alpha+\beta$ with irreducible representations $V_0, V_1, \mathrm{Sym}^k(V_1),$ and $\Lambda^2(V_1)$.

The determinant representation $\Lambda^2(V_1)$ is one-dimensional, and taking the tensor power of $\Lambda^2(V_1)$ with itself will give another one-dimensional representation (that is irreducible) with highest weight $2\alpha + 2\beta$. In fact, tensoring $\Lambda^2(V_1)$ repeatedly with itself will give all of the irreducible representations $\Lambda^k(V_1)$ with highest weight $k\alpha + k\beta$ for any positive integer k. But since the determinant is nonzero for $\mathrm{U}(2)$, we can actually obtain the irreducible representations $\Lambda^k(V_1)$ for *any* $k \in \mathbb{Z}$.

This is enough to classify *all* of the irreducible representations of U(2) (theorem 8.11 and corollary 8.12). Since $\Lambda^k(V_1)$ is one-dimensional (and hence irreducible), we may tensor $\Lambda^k(V_1)$ together with any of the $\mathrm{Sym}^k(V_1)$'s to get another irreducible representation.

Example 8.14. The group $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ has no roots, since the adjoint representation acting on $\mathfrak{u}(1)$ is trivial. The integer lattice and the weight lattice are just the integers. Each weight is the highest weight since there is no group action. Hence there is a one-to-one correspondence between each of the weights and the irreducible representations (by 8.8).

V_0 is the trivial representation; $\chi_{V_0} = 1$. $V_1 = \mathbb{C}$ is the defining representation. Since V_1 is one-dimensional, its character is V_1 itself: $\chi_{V_1} = e^{2\pi i\alpha}$. Furthermore, The tensor products $V_1^{\otimes n}$ for $n \in \mathbb{Z}$ are also one-dimensional (and hence irreducible). $\chi_n = e^{2\pi i n\alpha}$. Thus the irreducible representations of U(1) are $V_n = V_1^{\otimes n} = \mathbb{C}$, where the group action is given by $g \cdot v = e^{2\pi i n\alpha} v$ for all $n \in \mathbb{Z}$.

Bibliography

[Ad] J. F. Adams, *Lectures on Lie Groups*, The University of Chicago Press, 1969.

[Ar] M. Artin, *Algebra*, Pearson; 2nd edition, 2011.

[BrDi] T. Bröcker, T. tom Diek, *Representations of Compact Lie Groups*, Springer-Verlag New York Inc., 1985.

[Ch] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, 1946.

[FuHa] W. Fulton, J. Harris, *Representation Theory*, Springer-Verlag New York Inc., 1991.

[dC] M. P. do Carmo, *Riemannian Geometry*, Birkhäuser Boston, 1992.

[Fo] G.B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, Inc., 1999.

[HoKu] K. Hoffman, R. Kunze, *Linear Algebra*, Prentice-Hall, 1971.

[GiPo] V. Guillemin, A. Pollack, *Differential Topology*, Prentice-Hall, 1975.